On the Riemann-Hilbert approach to the asymptotic analysis of the correlation functions of the Quantum Nonlinear Schrödinger equation. Non-free fermionic case.

A. R. Its[1] and N. A. Slavnov[2]

[1] *Department of Mathematical Sciences IUPUI*
402 N. Blackford st., Indianapolis IN 46202-3216
itsa@math.iupui.edu

[2] *Steklov Mathematical Institute,*
Gubkina 8, 117966, Moscow, Russia
nslavnov@mi.ras.ru

Abstract

We consider the local field dynamical temperature correlation function of the Quantum Nonlinear Schrödinger equation with the finite coupling constant. This correlation function admits a Fredholm determinant representation. The related operator-valued Riemann–Hilbert problem is used for analysing the leading term of the large time and long distance asymptotics of the correlation function.

1 Introduction

This work continues the study of the correlation functions of the Quantum Nonlinear Schrödinger equation (QNLS) out off free fermionic point which was originated in the papers [1]–[5].

We consider the temperature correlation function of the local fields of the QNLS equation,

$$\langle\Psi(0,0)\Psi^\dagger(x,t)\rangle_T = \frac{\mathrm{tr}\left(e^{-\frac{H}{T}}\Psi(0,0)\Psi^\dagger(x,t)\right)}{\mathrm{tr}\, e^{-\frac{H}{T}}}. \tag{1.1}$$

Here $\Psi(x,t),\ \Psi^\dagger(x,t),\ (x,t\in\mathbf{R})$ are the canonical Bose fields obeying the standard equal-time commutation relations

$$[\Psi(x,t),\Psi^\dagger(y,t)] = \delta(x-y), \tag{1.2}$$

and acting in the Fock space as

$$\Psi(x,t)|0\rangle = 0, \qquad \langle 0|\Psi^\dagger(x,t) = 0. \tag{1.3}$$

The evolution of the field Ψ with respect to time t is usual

$$\Psi(x,t) = e^{iHt}\Psi(x,0)e^{-iHt}. \tag{1.4}$$

The Hamiltonian of the model H in (1.1) and (1.4) is equal to

$$H = \int dx \left(\partial_x \Psi^\dagger(x)\partial_x\Psi(x) + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) - h\Psi^\dagger(x)\Psi(x)\right). \tag{1.5}$$

Here $0 < c < \infty$ is the coupling constant and h is the chemical potential. The parameter T in (1.1) is a temperature.

The quantum Nonlinear Schrödinger equation describes one-dimensional Bose gas with delta-function interactions. The basic thermodynamic equation of the model is the Yang–Yang equation [6] for the energy of an one-particle excitation $\varepsilon(\lambda)$ in thermal equilibrium

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2}\ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right)d\mu. \tag{1.6}$$

The function $\varepsilon(\lambda)$ behaves as $\lambda^2 - h$ at $\lambda \to \infty$. It is positive for $\lambda \in R$, if $h < 0$, and it has two real roots $\varepsilon(\pm q) = 0$, if $h > 0$.

It is worth mentioning also the integral equation for the total spectral density of vacancies in the gas $\rho_t(\lambda)$:

$$2\pi\rho_t(\lambda) = 1 + \int_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2}\vartheta(\mu)\rho_t(\mu)\,d\mu, \tag{1.7}$$

where

$$\vartheta(\lambda) = \left(1+\exp\left[\frac{\varepsilon(\lambda)}{T}\right]\right)^{-1} \tag{1.8}$$

is the Fermi weight. The value $\vartheta(\lambda)\rho_t(\lambda)$ defines the spectral density of particles in the gas. Due to the properties of $\varepsilon(\lambda)$ the Fermi weight $\vartheta(\lambda)$ decays as $\exp\{-\lambda^2/T\}$ at $\lambda \to \infty$.

In this work we analyse a large time and long distance behavior of the correlation function (1.1).

The asymptotic evaluation of the correlation functions is one of the most challenging analytic problems in the theory of exactly solvable quantum field models. For the case of zero temperature the leading asymptotic term can be found via conformal field theory [7]. The small temperature limit also can be considered in the framework of this approach, although, strictly speaking, increasing temperature destroys conformal properties of the model. In the present paper we develop the method which is based on the Fredholm determinant representations of the correlation functions and which allows to remove the small temperature restriction.

A systematic exposition of the determinant representation method is given in [8]. For the reader's convinience, we shall outline the principal features of the scheme together with a brief historical review concerning the asymptotic analysis of the correlation functions.

The determinant representation method is based on the remarkable fact that the correlation functions of the 1+1 exactly solvable quantum models can be represented as Fredholm determinants of the integral operators V acting in $L^2(\Sigma, d\lambda)$ and whose kernels, $V(\lambda_1, \lambda_2)$, have the following special form :

$$V(\lambda_1, \lambda_2) = \frac{\sum_{j=1}^{N} e_j(\lambda_1) f_j(\lambda_2)}{\lambda_1 - \lambda_2}, \qquad \sum_j e_j(\lambda) f_j(\lambda) = 0, \tag{1.9}$$

where functions $e_j(\lambda), f_j(\lambda)$, and, in fact, the oriented contour of integration Σ, depend on the model under consideration. The first representation of this type was obtained in [9] for equal-time correlation functions in one-dimensional impenetrable ($c = \infty$) bosons. Later on, determinant formulae were derived for a majority of exactly solvable statistical mechanics and quantum field models (for the principal references we refer the reader to the book [8]). In particular, the Fredholm determinant for the time-dependent correlations in one-dimensional impenetrable bosons was constructed in [10]. A generalization of the results of [9] for non-free fermionic case was obtained in [11]. The determinant representation for the most general case of non - free fermionic time-dependent temperature correlation function (1.1) was found in [1].

The determinant formulae can be used to obtain nonlinear differential equations for quantum correlation functions. These nonlinear equations turn out to be *classical* integrable systems. Mor exactly, zero-temperature and equal-time two-point correlators are described by integrable ODEs of the Painlevé type (see [12] – [16]), while time-dependent and temperature or/and multi - point correlation functions appear to be the τ-functions of integrable PDEs (see [15], [17] – [20]). It is interesting to notice (see e.g. [21], [22], and section 9 of this paper) that if the quantum system is a result of quantization of a classical integrable system, as is the case for the QNLS model, then the integrable PDEs describing the correlation functions belong to the underlying classical integrable hierarchy.

The determinant representations for correlation functions can also be used to study their asymptotics. In the zero-temperature case, a comprehensive asymptotic analysis of the correlation functions related to XXO and impenetrable Bose gas models has been carried out in [23], [24], [16], [17], [25]. For the two-dimensional Ising model the analogous results were obtained earlier in [14] (see also [26] and [27]).

A further development of the determinant representation method was achieved in the series of papers [28], [18] (see also [8]). The approach of [28], [18] is based on the use of the Riemann-Hilbert method of the theory of *classical* integrable systems for the asymptotic evaluation of the Fredholm determinants describing the correlation functions of the *quantum* integrable systems. The Riemann-Hilbert method allowed to extend the mentioned above zero-temperature results for the XXO and impenetrable Bose gas models to the general finite-temperature case (see [28], [29], [20], and [30]).

The Riemann-Hilbert asymptotic scheme is the principal analytic tool which is used in the present paper. In what follows, we describe its basic ideas in some details.

Riemann-Hilbert problem (RHP) appears in the theory of correlation functions due to a simple yet important fact that the resolvent kernel corresponding to kernel (1.9) can be explicitly evaluated in terms of the solution of the matrix Riemann-Hilbert problem with the jump matrix $G(\lambda)$ given by the equation (cf. [28], [18], [21]):

$$G_{jk}(\lambda) = \delta_{jk} + 2\pi i e_j(\lambda) f_k(\lambda).$$

More exactly, let $\chi(\lambda)$ be a $N \times N$ matrix function which solves the following *Riemann-Hilbert* problem:

$$1.\ \chi(\lambda) \to I, \quad \lambda \to \infty, \quad \text{(normalization condition)} \tag{1.10}$$

$$2.\ \chi(\lambda) \quad \text{is analytic function of } \lambda \text{ if } \lambda \notin \Sigma, \tag{1.11}$$

$$3.\ \chi_-(\lambda) = \chi_+(\lambda) G(\lambda), \qquad \lambda \in \Sigma, \qquad \text{(jump condition)}, \tag{1.12}$$

where $\chi_\pm(\lambda)$ denote the $(\pm)$ - boundary values of the function $\chi(\lambda)$ on Σ, i.e.

$$\chi_\pm(\lambda) = \lim_{\lambda' \to \lambda} \chi(\lambda'), \quad \lambda' \in (\pm) - \quad \text{side of} \quad \Sigma.$$

Then, the resolvent kernel $R(\lambda_1, \lambda_2)$ corresponding to the kernel (1.9) $(1 - R = (1+V)^{-1})$ is given by the following explicit formulae,

$$R(\lambda_1, \lambda_2) = \frac{\sum_{j=1}^{N} E_j(\lambda_1) F_j(\lambda_2)}{\lambda_1 - \lambda_2}, \tag{1.13}$$

$$E_j(\lambda) = \sum_{k=1}^{N} (\chi_+(\lambda))_{jk} e_k(\lambda), \quad F_j(\lambda) = \sum_{k=1}^{N} f_k(\lambda) (\chi_+^{-1}(\lambda))_{kj}.$$

The dynamical parameters, i.e. distance x and time t, enter the jump matrix $G(\lambda)$ through the transformation,

$$G(\lambda) \to e^{D(\lambda;x,t)} G(\lambda) e^{-D(\lambda;x,t)}, \tag{1.14}$$

where the rational in λ and linear in x, t diagonal matrix function $D(\lambda; x, t)$ represents the dispersion law of the underlying classical model (e.g. $D(\lambda; x, t) = \mathrm{diag}(it\lambda^2 - ix\lambda, \quad -it\lambda^2 + ix\lambda)$ for the NLS equation). It can be shown (see e.g. [8]; see also [31]) that the logariphmic derivatives of $\det(1+V)$ with respect to x and t (and, in fact, with respect to any other physical parameter) are explicitly expressible in terms of the resolvent R. Hence the asymptotic analysis of the original Fredholm determinat is reduced by (1.13) to the asymptotic analysis of the *oscillatory* Riemann-Hilbert problem (1.10-1.12, 1.14).

In the theory of classical integrable systems, the Riemann-Hilbert problems of the type (1.10-1.12, 1.14) represent solutions of the Cauchy problems for integrable PDEs. In this context, the development of the relevant apparatus for the asymptotic analysis of the oscillatory matrix RHPs

was originated in 1973-1977 in the works [32] – [36]. It was essentially completed (for a detailed historical review see [37]) in 1993 in the paper [38] where a nonlinear analog of the classical steepest descent method for oscillatory Riemann-Hilbert problems was suggested.

The Deift-Zhou nonlinear steepest descent method consists of three basic steps (see [38]; see also [39] and [37]):

1. A deformation of the original jump contour Σ to the steepest descent (with respect to the dispersion exponent $D(\lambda; x, t)$) contours and asymptotic evaluation of the solution away from the corresponding saddle points.

2. The use of the relevant Lax pair and certain model Riemann-Hilbert problems to construct a parametrix for the solution near the saddle points.

3. Assembling the above pieces into a *uniform* asymptotic solution which makes it possible to justify the whole construction by standard estimates [40] of the theory of singular integral operators on the complicated contours.

Each of these steps has its natural analog in the classical steepest descent method for oscillatory contour integrals yet exploits much more sophisticated technics and analysis. In contrast with the classical steepest descent method, which is used for asymptotic evaluation of oscillatory *integral representations*, the nonlinear steepest descent method deals with a special type of oscillatory (singular) *integral equations.* In fact, the RHP (1.10-1.12) is equivalent to the following singular integral equation:

$$\chi_+(\lambda) = I + \frac{1}{2\pi i}\int_\Sigma \chi_+(\mu)(I - G(\mu))\frac{d\mu}{\mu - \lambda_+}\,. \tag{1.15}$$

The nonlinear steepest descent method provides a regular way of finding the proper transformation (which is highly nontrivial and virtually impossible to be seen directly!) of the original singular equation (1.15) to an equivalent one with uniformly small kernel.

Some of the principal ideas involved in the steps 1, 2 of the nonlinear steepest descent method (e.g. explicit solutions for the model Riemann-Hilbert problems) go back to the earlier works [33] and [41]. These earlier versions of the Riemann-Hilbert approach had already been successfully exploited in the asymptotic analysis of the various temperature correlation functions (see [28], [29], [20]). The use of the nonlinear steepest descent method increases considerably the power of the original scheme of [28], [18]. In partricular, in [30] the use of the nonlinear steepest descent method allowed to compute the long-time asymptotics of the autocorrelation function of the transverse Ising chain at the critical magnetic field for the first time at finite temperature. The method has also produced solutions for some long-standing problems in the theory of random matrices and orthogonal polynomials (see e.g. [42], [43], and [44]).

The XXO magnet and impenetrable Bose gas are free fermionic models. The non-free fermionic case is much more complicated because of several reasons. First, the Riemann-Hilbert problems,

which describe the corresponding Fredholm determinants, become operator-valued (cf. [8], [3]): the matrix elements $G_{j,k}(\lambda)$ turn into integral operators $\hat{G}_{j,k}(\lambda)$ acting in an auxilary L_2 space ($G_{j,k}(\lambda) \to G_{j,k}(\lambda|u,v)$). In other words, an infinite-dimensional environment, absent in the free fermionic problems, arises. This transforms the associated Lax pairs and classical nonlinear PDEs into their non-Abelian analogies and obviously provides new significant difficulties for the asymptotic analysis. In fact, up to now the only attempt to solve an operator-valued RHP, which is related to correlation function of XXZ magnet, had been done in [45].

The second difficulty is more subtle than the first one, and it is related to the presence of another infinite-dimensional context in the non-free fermionic problems; this time – of the quantum field nature. The fact of the matter is that out off free fermionic point the correlation functions can not be presented directly in the determinant form. Due to existence of non-trivial S-matrix, one needs to introduce [46] auxiliary quantum operators — Korepin's dual fields — in order to find such a representation. As a result, the determinants obtained depend on these operators, and the correlation functions are equal to the vacuum expectation values of the Fredholm determinants in an auxiliary Fock space. In particular, for the correlation function (1.1) the following equation takes place (cf. [1]),

$$\langle \Psi(0,0)\Psi^\dagger(x,t)\rangle_T = const \cdot e^{-iht}(0|\mathcal{B}\Big([\psi,\phi_A,\phi_D],x,t\Big)|0)\,, \tag{1.16}$$

where the factor *const* only depends on the temperature, coupling constant and chemical potential, and $\mathcal{B}$ is an operator acting in the auxilary Fock space with the vacuum vector $|0)$. It involves the Fredholm determinant which functionally depends on the three basic quantum operators (dual fields) $\psi(\lambda), \phi_A(\lambda)$, and $\phi_D(\lambda)$. The exact definitions of the quantum operator $\mathcal{B}$ is given in appendix A. The dual fields $\psi(\lambda), \phi_A(\lambda), \phi_D(\lambda)$ are defined in section 2 by equations (2.22) - (2.23).

The operator-valued RHPs allow to evaluate the Fredholm determinants, but not their vacuum mean values. The two infinite-dimensional contexts, i.e. the operator nature of the RHP and the presence of the dual quantum fields, are completely unrelated. Therefore, in order to find the large time and long distance asymptotics of a non-free fermionic correlation function one needs to be sure that the asymptotics of the mean value is equal to the mean value of the asymptotics.

It had been shown in [4] that a naive asymptotic analysis of the Fredholm determinants containing dual fields does not provide a satisfactory result, i.e. the asymptotics is not uniform with respest to the avaraging over the dual fields. Thus, the problem arises — to obtain the asymptotic description of the determinant related to (1.1) which would be stable with respect to the procedure of averaging over the dual fields. This is the problem which we deal with in the present paper.

Let us describe now the content of the paper.

In section 2, following [1] - [3], we give the basic formulæ and definitions concerning the non-free fermionic version of the determinant representation – Riemann-Hilbert approach to the asymptotic

analysis of the correlation function (1.1).

In section 3, following [3], we formulate the central object of the analysis, i.e. the operator-valued RHP associated with the correlation function (1.1) (see (3.1) and (3.11) below). The jump contour Σ of the RHP coincides with the real line, and the corresponding jump operator $\hat{G}(\lambda)$ is realized as a 2×2 matrix whose entries $\hat{G}_{jk}(\lambda)$ are the integral operators in an auxilary $L_2(-\infty, \infty)$ space. A principal feature of this RHP is that the operators $\hat{G}_{jk}(\lambda)$ have the following special structure:

$$\hat{G}_{jk}(\lambda) = \hat{\imath}\, \delta_{jk} + |j\rangle (G_{jk} - \delta_{jk}) \langle k|, \quad j, k = 1, 2, \tag{1.17}$$

where $\hat{\imath}$ is the identical operator in $L_2(-\infty, \infty)$, i.e. its kernel is a delta - function: $\hat{\imath}(u, v) = \delta(u - v)$. The symbols $|j\rangle \equiv |j, u\rangle$ and $\langle k| \equiv \langle k, v|$ denote certain elements of $L_2(-\infty, \infty)$ (see definition (3.9) below) satisfying the normalization condition,

$$\langle 1|1\rangle \equiv \int_{-\infty}^{\infty} \langle 1, u|1, u\rangle\, du = \langle 2|2\rangle \equiv \int_{-\infty}^{\infty} \langle 2, u|2, u\rangle\, du = 1. \tag{1.18}$$

The numerical matrix $G(\lambda)$ in (1.17) is closely related to the jump matrix of the free fermionic impenetrable Bose gas model (cf. [29]).

Equations similar to (1.17) take place for other non-free fermionic correlation functions as well, and they have a very important algebraic meaning. In fact, as it can be easiely verified, formulae (1.17) and (1.18) define a representation,

$$G \to \hat{G} \equiv \hat{\mathcal{A}}(G), \tag{1.19}$$

of the group $\mathbf{GL}(2, \mathrm{C})$ in the group of Fredholm invertible operators in $L_2((-\infty, \infty), \mathrm{C}^2)$. It also can be shown (see [47]) that representation (1.19) preserves determinants, i.e.

$$\det G = \det \hat{G}. \tag{1.20}$$

The link (1.19) between the non-free fermionic jump operators and their free fermionic counterparts was first noticed by Korepin, and it has already proved very usefull in the asymptotic analysis of the non-free fermionic correlators (see [47] and [45]). The mapping (1.19) plays a crucial role in the following sections where we develop the operator-valued version of the Deift-Zhou nonlinear steepest descent method.

The main obstacle in taking the full advantage of the group-representation nature of equation (1.17) is that the vectors $|j\rangle$ and $\langle k|$ *depend on the parameter* λ, i.e. $|j\rangle \equiv |j, u, \lambda\rangle$ and $\langle k| \equiv \langle k, v, \lambda|$ (see (3.9) below); moreover, they become singular for complex λ. Therefore, one can not solve a non-free fermionic (operator) Riemann-Hilbert problem by just applying the representation (1.19) to the solution of the corresponding free fermionic (matrix) Riemann Hilbert problem. Nevertheless, this representation helps to perform the first step of the nonlinear steepest descent

method (the contour deformation) and evaluate the leading term of the asymptotic solution of the RHP. This is done in sections 4, 5 where we closely follow the methodology of [45]. The leading asymptotic term we found coincide with the one obtained earlier in [4] via the direct asymptotic analysis of the Fredholm determinant.

The objective of sections 6–8 is an order $t^{-1/2}$ correction. This is the second step of the nonlinear steepest descent method. In the free-fermionic case (see e.g. [39] and [37]) this step constitutes the reduction of the deformed RHP to a model problem associated with the saddle point. The model problem is then solved explicitly in terms of the parabolic cylinder functions (see again [39] and [37]; see also [41]). In our case, the model problem is an operator-valued version of the classical model problem. In fact, the jump operator of the model problem is the image of a certain jump matrix, closely related to the corresponding free fermionic model jump matrix, under the *modified* mapping $\hat{\mathcal{A}}_0$. The latter is the representation $\hat{\mathcal{A}}$ with the vectors $|j\rangle$ and $\langle k|$ replaced by their values evaluated at the saddle point λ_0,

$$| \mathring{j}, u\rangle = |j, u, \lambda_0\rangle, \qquad \langle \mathring{k}, v| = \langle k, v, \lambda_0|. \tag{1.21}$$

The representation $\hat{\mathcal{A}}_0$ does not depend on λ and hence can be used to transform a free fermionic matrix model solution into the operator model solution. The transfortmation though is not quite straightforward, but it allows eventually (see sections 7, 8) to solve the model RHP explicitly (more exactly, up to the inversion of an integral operator which is independent on λ, x, and t). In achieving this result, the following 'operator - indexed' generalization $D_{\hat{\nu}}(\xi)$ of the classical parabolic cylinder functions $D_\nu(\xi)$,

$$D_{\hat{\nu}}(\xi) = D_0(\xi)\left(\hat{\imath} - | \mathring{j}\rangle\langle \mathring{j} \,|\right) + | \mathring{j}\rangle\langle \mathring{j} \,| D_\nu(\xi), \qquad \hat{\nu} = \nu| \mathring{j}\rangle\langle \mathring{j} \,|, \tag{1.22}$$

plays an important role.

The order $t^{-1/2}$ correction is a main threshold in the asymptotic analysis of the temperature correlation functions (cf. e.g. [29]). After it is passed, one can, in principle, obtain a total asymptotic expansion using the elementary algebra only, just substituting a proper asymptotic series into the nonlinear PDEs associated with the correlation function under consideration. In our case, the relevant nonlinear PDE is the mentioned above non-Abelian version of the classical NLS equations (see (2.21) below). In section 9 we use this non-Abelian NLS for exact evaluation of a few terms next to the order $t^{-1/2}$ term and for evalution of the pre-exponential factor in the asymptotics of the Fredholm determinant. The non-Abelian NLS also allow us to extract a valuable information about the full asymptotic expansion. We analyse this information in section 10 and conclude that to make the asymptotic expansion obtained stable with respect to the dual field avaraging a certain modification of our construction is needed. The modification concerns the equation determing the saddle point λ_0, and it is made in the last section 11.

As the **main result** of the paper, we suggest the following asymptotic formulae for the quantum operator $\mathcal{B}$ in the r.h.s. of equation (1.16),

$$\mathcal{B} = C_-(\phi_D, \phi_A|\lambda_0, T, h, c)(2t)^{-\frac{(\nu(\Lambda)+1)^2}{2}} e^{\psi(\Lambda)+it\Lambda^2-ix\Lambda}$$

$$\times \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(x-2\lambda t+i\psi'(\lambda)\Big)\,\mathrm{sign}(\Lambda-\lambda)\right.$$

$$\left.\times \ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\Lambda)}\Big)\Big\}\,d\lambda\right\}\left(1+\mathcal{O}\left(\frac{\ln^2(t)}{t}\right)\right), \tag{1.23}$$

$$t\to\infty, \qquad \frac{x}{2t}\equiv\lambda_0=O(1),$$

for the case of negative chemical potential h, and

$$\mathcal{B} = C_+(\phi_D, \phi_A|\lambda_0, T, h, c)(2t)^{-\frac{\nu^2(\Lambda)}{2}} e^{\psi(\Lambda_1)+it\Lambda_1{}^2-ix\Lambda_1}$$

$$\times \exp\left\{\frac{1}{2\pi}\int_{\Gamma}\Big(x-2\lambda t+i\psi'(\lambda)\Big)\,\mathrm{sign}(\Lambda-\Re\lambda)\right.$$

$$\left.\times \ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\,\mathrm{sign}(\Re\lambda-\Lambda)}\Big)\Big\}\,d\lambda\right\}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{t}}\right)\right), \tag{1.24}$$

$$t\to\infty, \qquad \frac{x}{2t}\equiv\lambda_0=O(1),$$

for the case of positive chemical potential h.

A few remarks explaining the notations which are used in equations (1.23), (1.24) and the meaning of the equations themselves are needed:

1. The function $\vartheta(\lambda)$ is the Fermi weight (1.8).

2. The dual field $\phi(\lambda)$ equals the difference $\phi_A(\lambda)-\phi_D(\lambda)$.

3. The exponent $\nu(\Lambda)$ is defined by the formula,

$$\nu(\Lambda) = -\frac{1}{2\pi i}\ln\Big[\Big(1-\vartheta(\Lambda)(1+e^{-\phi(\Lambda)})\Big)\Big(1-\vartheta(\Lambda)(1+e^{\phi(\Lambda)})\Big)\Big],$$

4. Λ_1 and Λ are the roots of the equations

$$1-\vartheta(\Lambda_1)(1+e^{-\phi(\Lambda_1)})=0, \quad h>0, \tag{1.25}$$

and

$$\Lambda = \lambda_0 + \frac{i}{2t}\psi'(\Lambda), \tag{1.26}$$

respectively. Since Λ is not necessary real one should understand the integrals (1.23), (1.24) as

$$\int_{-\infty}^{\infty} F\Big(\mathrm{sign}(\Lambda-\lambda),\lambda\Big)\,d\lambda = \int_{-\infty}^{\Lambda} F(1,\lambda)\,d\lambda + \int_{\Lambda}^{\infty} F(-1,\lambda)\,d\lambda. \tag{1.27}$$

5. The undetermined factors $C_{\pm}(\phi_A,\phi_D|\lambda_0,T,h,c)$ are functionals of the dual fields ϕ_A, ϕ_D, and they are functions of the temperature T, the chemical potential h, and the coupling constant c. They do not depend on the dual field ψ, and they only depend on distance x and time t through the ratio $\lambda_0 = x/2t$.

6. All the commutation relations involving the dual fields $\phi_A(\lambda)$, $\phi_D(\lambda)$, and $\psi(\lambda)$ are trivial (see (2.24)). Therefore, when deriving (1.23) and (1.24) we treat the dual fields as the complex-valued functions analytic in the strip $|\operatorname{Im}\lambda| < c/2$; the functions $\exp\phi_{A,D}$ and $\exp\psi$ are supposed to be bounded in the strip. It is also assumed that equation (1.25) has no real solutions if $h<0$, and it has exactly two real roots, i.e. Λ_1, Λ_2, if $h>0$. Moreover, it is supposed that
$$\Lambda_1 < \Lambda_2 < \Lambda, \tag{1.28}$$
and that the same inequalities, in the case $h>0$, take place for the roots of the function $1-\vartheta(\lambda)(1+e^{\phi(\lambda)})$. These assumptions are justified in sections 2, 3 and 5. Accordingly, the contour of integration Γ in (1.24) is choosen as it is indicated in Figure 2 below, so that the integral in (1.24) makes sense.

7. The point Λ, which plays the role of the 'shifted' saddle point, is defined by equation (1.26) as its root which approches λ_0 as $t\to\infty$. In other words, $\Lambda-\lambda_0 = \mathcal{O}(t^{-1})$. Nevertheless, we do not replace Λ by λ_0 in the asymptotic formulae (1.23 - 1.24). As it is explained in sections 10 and 11, to make the asymptotics of $\mathcal{B}$ compatible with the avaraging over the dual fields we need to trace the presence of the field ψ in all the corrections to the leading terms indicated in (1.23 - 1.24). In fact, we need all the corrections to be of the order $\mathcal{O}(\psi^m/t^n)$ with $0<m<n$. The replacement $\Lambda\mapsto\lambda_0$, if made, would introduce the corrections of the order $\mathcal{O}(\psi^m/t^n)$ with $m\geq n$ which produce the positive powers of t after the avaraging in (1.16) (see sections 10 and 11 for more details). In other words, when the operator nature of equation (1.26) is taking into account one has to keep in mind that, although the factor $i/2t$ is small, the quantity $\psi'(\lambda)$ is an unbounded operator of the derivative type.

8. Because of the quantum operator nature of the dual fields $\phi_A(\lambda)$, $\phi_D(\lambda)$, and $\psi(\lambda)$, the r.h.s. of equations (1.23) and (1.24) have, strictly speaking, a symbolic meaning. One needs the method of avaraging of the functionals of the dual fields developed in [5] in order to evaluate

the mean values of the objects indicated in (1.23 - 1.24). The results of these calculations, which would finalize the asymptotic analysis of the leading terms of the correlation function (1.1), will be given in the forthcoming paper.

We conclude the introduction by emphasizing that in this paper we do not address the questions related to the third step of the nonlinear steepest descend method, i.e. the questions related to the rigorous justification of the asymptotic equations (1.23 - 1.24). Some of these questions, including the rigorous setting and the general theory of the operator-valued Riemann- Hilbert problems arising in the non-free fermionic models, are still under the investigation. The purpose of this paper is to show that, using the operator-valued Riemann-Hilbert problems, it is posible to carry out an effective asymptotic analysis of the general time-dependent temperature non-free fermionic correlation functions based on their determinant representation.

2 The basic formulæ and definitions

This sections contains the basic definitions and notations, which are used in the paper. The reader may find the details in the papers [1], [3]. Here we restrict our selves with only necessary formulæ.

Consider the operators of the following form

$$\hat{A} \equiv \hat{A}(u,v) = \begin{pmatrix} \hat{A}_{11}(u,v) & \hat{A}_{12}(u,v) \\ \hat{A}_{21}(u,v) & \hat{A}_{22}(u,v) \end{pmatrix} \tag{2.1}$$

The entries of this 2×2 matrix are integral kernels $A_{jk}(u,v)$ acting on functions from the $L_2(-\infty,\infty)$ space. We shall mark these operators (as well as their entries) with the symbol 'hat'. Also, we will frequently use the equation,

$$\hat{M} \equiv \hat{M}(u,v),$$

to indicate that $\hat{M}(u,v)$ is the kernel of an operator $\hat{M}$.

The product of two operators of this type is defined as

$$(\hat{A}\hat{B})_{jk}(u,v) = \sum_{l=1}^{2} \int_{-\infty}^{\infty} \hat{A}_{jl}(u,w)\hat{B}_{lk}(w,v)\,dw. \tag{2.2}$$

We shall also consider the products of separate matrix elements:

$$(\hat{A}_{jk}\hat{B}_{lm})(u,v) = \int_{-\infty}^{\infty} \hat{A}_{jk}(u,w)\hat{B}_{lm}(w,v)\,dw. \tag{2.3}$$

The definition of trace of 'hat'-operators is standard

$$\mathrm{Tr}\,\hat{A} = \mathrm{tr}\,\hat{A}_{11} + \mathrm{tr}\,\hat{A}_{22} = \int_{-\infty}^{\infty} \left(\hat{A}_{11}(u,u) + \hat{A}_{22}(u,u)\right)\,du, \tag{2.4}$$

where

$$\operatorname{tr} \hat{A}_{jk} = \int_{-\infty}^{\infty} \hat{A}_{jk}(u,u)\, du. \tag{2.5}$$

The main object of study in this paper is an operator-valued Riemann–Hilbert problem. Here we give the general formulation of this problem.

The RHP consists of the following: find an operator $\hat{\chi}(\lambda)$ depending on complex parameter λ and satisfying the following conditions (cf. (1.10) – (1.12)):

$$1^{(0)}.\ \hat{\chi}(\lambda) \to \hat{I}, \quad \lambda \to \infty, \quad \text{(normalization condition)} \tag{2.6}$$

$$2^{(0)}.\ \hat{\chi}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R, \tag{2.7}$$

$$3^{(0)}.\ \hat{\chi}_-(\lambda) = \hat{\chi}_+(\lambda)\hat{G}(\lambda), \qquad \lambda \in R, \qquad \text{(jump condition)}. \tag{2.8}$$

Here $\hat{I}$ is the identity operator

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(u-v). \tag{2.9}$$

Operators $\hat{\chi}_\pm(\lambda)$ are boundary values of the function $\hat{\chi}(\lambda)$ from the left (from up half-plane) and from the right (from low half plane) of the real axis respectively (compare again with (1.10) – (1.12)). $\hat{G}(\lambda)$ is a given operator. A particular choice of this operator which corresponds to the correlation function (1.1) will be described in detail in the next section.

Following the tradition we call the operator $\hat{G}(\lambda)$ 'jump matrix', although this matrix evidently is infinite dimensional. We hope that this terminology will not cause a confusion.

Although the RHP (0) looks like 2×2 matrix RHP, here we deal with infinite-dimensional integral operators. In more detail, the jump condition $3^{(0)}$ can be written as

$$\hat{\chi}_{jk-}(\lambda|u,v) = \sum_{l=1}^{2} \int_{-\infty}^{\infty} \hat{\chi}_{jl+}(\lambda|u,w)\hat{G}_{lk}(\lambda|w,v)\, dw, \qquad \lambda \in R. \tag{2.10}$$

The analyticity of the operator $\hat{\chi}(\lambda)$ is understood as the λ-analyticity of its kernel $\hat{\chi}(\lambda|u,v)$. All the limits and expansions involving $\hat{\chi}(\lambda) \equiv \hat{\chi}(\lambda|u,v)$ are supposed to be uniform with respect to u and v. We will not go further in the specification of the setting of the RHP (2.6) – (2.8). As it has already been indicated in the introduction, in this paper we do not discuss the general theory of the operator-valued RHPs. In what follows, we assume that the unique solutions of the operator-valued RHPs we are dealing with always exist and satisfy all the natural properties. For instance, we shall assume that if $\hat{G}(\lambda)$ is a Fredholm operator, i.e. the determinant $\det \hat{G}(\lambda)$ exists, then $\hat{\chi}(\lambda)$ is a Fredholm operator, and its determinant $\det \hat{\chi}(\lambda)$ is a unique solution of the following *scalar* RHP,

$$1.\ \det \hat{\chi}(\lambda) \to 1, \quad \lambda \to \infty, \tag{2.11}$$

$$2.\ \det \hat{\chi}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R, \tag{2.12}$$

$$3.\ \det\hat\chi_-(\lambda)=\det\hat\chi_+(\lambda)\det\hat G(\lambda),\quad \lambda\in R. \tag{2.13}$$

In particular, this means that the function $\det\hat\chi(\lambda)$ is given by the explicit formula,

$$\det\hat\chi(\lambda)=\exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\ln\det\hat G(\mu)\right\}, \tag{2.14}$$

subject the zero-index solvability condition,

$$\arg\det\hat G(\lambda)\to 0,\quad \lambda\to\pm\infty\,. \tag{2.15}$$

The asymptotic expansion of the solution of the RHP at $\lambda\to\infty$ will play the central role

$$\hat\chi(\lambda)=\hat I+\frac{\hat b}{\lambda}+\frac{\hat c}{\lambda^2}+\dots\ . \tag{2.16}$$

The first two coefficients of this expansion $\hat b$ and $\hat c$ allow one to reconstruct the Fredholm determinant, i.e the τ-function corresponding to the solution $\hat\chi(\lambda)$, which in turn directly related to the correlation function of the local fields (1.1). This correlation function can be presented as

$$\langle\Psi(0,0)\Psi^\dagger(x,t)\rangle_T=const\cdot e^{-iht}(0|\mathcal{B}\Big([\psi,\phi_A,\phi_D],x,t\Big)|0). \tag{2.17}$$

We have extracted in this formula the factor $\exp\{-iht\}$, which trivially depends on t, and a constant, which does not depend on x and t. Strictly speaking this 'constant' depends on the temperature, coupling constant and chemical potential. The most important object in (2.17) is $\mathcal{B}\Big([\psi,\phi_A,\phi_D],x,t\Big)$, which, in particular, includes a Fredholm determinant, and which is an operator in auxiliary Fock space. This operator is a function of x and t and it functionally depends on three quantum operators (dual fields) $\psi(\lambda)$, $\phi_A(\lambda)$ and $\phi_D(\lambda)$. The last ones act in the auxiliary Fock space having the vacuum vector $|0)$ and dual vector $(0|$ (one should not confuse the auxiliary vacuum vector $|0)$ with the vector $|0\rangle$ in (1.3)). Thus, the correlation function is proportional to the vacuum expectation value of the operator $\mathcal{B}$. The detailed definition and the properties of the dual fields will be given later in this section. First we want to focus our attention on the relationship between the operator $\mathcal{B}$ and the operator-valued RHP.

The following representation for the operator $\mathcal{B}$ had been found in [1], [2]

$$\mathcal{B}\Big([\psi,\phi_A,\phi_D],x,t\Big)=\det\left(\tilde I+\widetilde V\right)\cdot\int_{-\infty}^{\infty}\hat b_{12}(u,v)\,dudv. \tag{2.18}$$

We call (2.18) 'determinant representation', since the r.h.s. of this formula is proportional to the Fredholm determinant of the linear integral operator $\tilde I+\widetilde V$. The detailed description of this integral operator is given in Appendix A.

The results of the paper [3] allow one to express the Fredholm determinant and factor $\hat{b}_{12}(u,v)$ in terms of the solution of an operator-valued RHP (0) with a certain jump matrix $\hat{G}(\lambda)$. In fact, the operator $\hat{b}_{12}$ in (2.18) is the corresponding entry of the operator coefficient $\hat{b}$ of the asymptotic expansion (2.16). The logarithmic derivatives of $\det\left(\tilde{I}+\widetilde{V}\right)$ with respect to distance x and time t also can be written in terms of the coefficients $\hat{b}$ and $\hat{c}$ from (2.16):

$$\begin{aligned} \partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= i\,\mathrm{tr}\,\hat{b}_{11}, \\ \partial_t \log\det\left(\tilde{I}+\widetilde{V}\right) &= i\,\mathrm{tr}(\hat{c}_{22}-\hat{c}_{11}). \end{aligned} \tag{2.19}$$

The second logarithmic derivatives depend on the 'matrix elements' $\hat{b}_{12}$ and $\hat{b}_{21}$ only:

$$\begin{aligned} \partial_x\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= -\,\mathrm{tr}(\hat{b}_{12}\hat{b}_{21}), \\ \partial_t\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= i\,\mathrm{tr}(\partial_x\hat{b}_{12}\cdot\hat{b}_{21}-\partial_x\hat{b}_{21}\cdot\hat{b}_{12}). \end{aligned} \tag{2.20}$$

Finally these two operators satisfy a non-Abelian generalization of the classical Nonlinear Schrödinger equation

$$\begin{aligned} -i\partial_t\hat{b}_{12} &= -\partial_x^2\hat{b}_{12}+2\hat{b}_{12}\hat{b}_{21}\hat{b}_{12}, \\ i\partial_t\hat{b}_{21} &= -\partial_x^2\hat{b}_{21}+2\hat{b}_{21}\hat{b}_{12}\hat{b}_{21}. \end{aligned} \tag{2.21}$$

Recall that one should understand the products of operators $\hat{b}_{12}$ and $\hat{b}_{21}$ in the r.h.s. of (2.20) and (2.21) in the sense (2.3)

Thus, we have described $\mathcal{B}$ in terms of the solution of the operator-valued RHP (2.6 - 2.8) (the explicit expression for the jump matrix $\hat{G}(\lambda)$ will be given in the next section). Equations (2.19) define $\mathcal{B}$ up to a constant factor. Equations (2.20), (2.21) will help us to analyse the higher corrections to the leading asymptotics of $\hat{b}_{12}$ and $\hat{b}_{21}$.

Let us turn now to the auxiliary quantum operators—dual fields. These operators $\psi(\lambda)$, $\phi_D(\lambda)$ and $\phi_A(\lambda)$ (originally the last two fields were denoted as $\phi_{D_1}(\lambda)$ and $\phi_{A_2}(\lambda)$) were introduced in [1] in order to remove two-body scattering and to reduce the model to free fermionic one. As it was mentioned already, they act in an auxiliary Fock space. Each of these fields can be presented as a sum of creation and annihilation parts

$$\begin{aligned} \phi_A(\lambda) &= q_A(\lambda)+p_D(\lambda), \\ \phi_D(\lambda) &= q_D(\lambda)+p_A(\lambda), \\ \psi(\lambda) &= q_\psi(\lambda)+p_\psi(\lambda). \end{aligned} \tag{2.22}$$

Here $p(\lambda)$ denotes the annihilation parts of the dual fields: $p(\lambda)|0)=0$; $q(\lambda)$ denotes the creation parts of the dual fields: $(0|q(\lambda)=0$.

The only nonzero commutation relations are

$$
\begin{aligned}
&[p_A(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_A(\mu)] = \ln h(\mu, \lambda), \\
&[p_D(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_D(\mu)] = \ln h(\lambda, \mu), \\
&[p_\psi(\lambda), q_\psi(\mu)] = \ln[h(\lambda, \mu)h(\mu, \lambda)], \quad \text{where} \quad h(\lambda, \mu) = \frac{\lambda - \mu + ic}{ic}.
\end{aligned}
\tag{2.23}
$$

Recall that c is the coupling constant in (1.5). It follows immediately from (2.23) that the dual fields belong to an Abelian sub-algebra

$$
[\psi(\lambda), \psi(\mu)] = [\psi(\lambda), \phi_a(\mu)] = [\phi_b(\lambda), \phi_a(\mu)] = 0, \tag{2.24}
$$

where $a, b = A, D$. Due to the property (2.24), the Fredholm determinant $\det\left(\tilde{I} + \widetilde{V}\right)$ is well-defined.

The Fredholm determinant $\det\left(\tilde{I} + \widetilde{V}\right)$ and the factor $\hat{b}_{12}$ in the representation (2.18) functionally depend on the dual fields. Hence $\mathcal{B}$ is an operator in auxiliary Fock space (for explicit formulæ see (A.1)–(A.6)). From the equations (2.19), (2.20) one can conclude, that the solution of the corresponding operator-valued RHP, as well as the jump matrix, also should depend on these auxiliary operators. We shall see in the next section that the jump matrix $\hat{G}(\lambda)$ does depend on dual fields, therefore it is an operator in the auxilary Fock space as well.

However, we would like to emphasize that the operator nature of the RHP (0) *is not related* to the fact that $\hat{\chi}(\lambda)$ and $\hat{G}(\lambda)$ depend on dual fields. Even if we replace all dual fields by some complex functions, the RHP (0) still remains the operator-valued one, since the entries of $\hat{\chi}(\lambda)$ and $\hat{G}(\lambda)$ are integral kernels.

The calculation of the asymptotics of the correlation function (1.1) consists of two stages. At the first stage one has to solve the operator-valued RHP (0) in order to find explicit expression for the operator $\mathcal{B}$. The second stage consists of the averaging with respect to auxiliary vacuum vectors. The role of dual fields is significantly different at these two stages. While the averaging is based essentially on the operator properties of dual fields, at the first stage the analytical properties of the last ones are much more important. While solving the RHP (0) one can consider the dual fields as some complex functions, which are holomorphic in a neighborhood of the real axis. Indeed, all these auxiliary operators commute with each other, and their matrix elements are holomorphic functions if their arguments belong to the strip $|\operatorname{Im}\lambda| < c/2$.

However such a treatment of dual fields leads us to a problem, which we had touched briefly in the Introduction. The matter is that the operator-valued RHP considered in the present paper can be solved only asymptotically for the large time t and long distance x. Thus, we obtain only asymptotic solution for RHP, and, hence, we find only the asymptotics of the operator $\mathcal{B}$. Strictly speaking, one has to be sure that vacuum expectation value of the asymptotics is equal to the asymptotics of the vacuum expectation value. Otherwise corrections to the asymptotic expression for $\mathcal{B}$ may give non-vanishing contribution into $(0|\mathcal{B}|0)$.

Below (section 11) we shall see that there exists a set of different asymptotic expansions of the operator $\mathcal{B}$, which are equivalent, if we treat the dual fields as complex functions. All of them provide us with the same result for the vacuum expectation value if and only if we take into consideration the complete asymptotic series. However, if we restrict our selves with the leading term of asymptotics and finite set of corrections, then these asymptotic series give different results for $(0|\mathcal{B}|0)$. In section 11 we will suggest an algorithm of choosing the asymptotic expansion which is compatible with the avaraging over the dual fields.

In summary we can formulate our point of view on the dual field problem as follows. We consider the dual fields as the complex functions, which are holomorphic in a finite neighborhood of the real axis. Moreover, since the matrix elements of $\exp\phi_a(\lambda)$ and $\exp\psi(\lambda)$ are rational functions of λ, we shall consider $\exp\phi_a(\lambda)$ and $\exp\psi(\lambda)$ as the rational functions of λ, bounded in the strip $|\operatorname{Im}\lambda| < c/2$. This permits us to solve the operator-valued RHP for the large time and long distance separation by deforming the original contour to the contours in the complex plane. At the same time we do not forget about the quantum operator nature of these auxiliary operators. Therefore we do not restrict our selves with the leading term of the asymptotics for the operator $\mathcal{B}$. We draw our special attention to the corrections and their behavior after averaging with respect to the auxiliary vacuum. We shall consider all these questions in more detail in sections 10, 11, where we will demonstrate how one can modify the procedure of solving the RHP in order to obtain correct result for the asymptotics of the vacuum mean value.

3 The operator-valued RHP

The operator-valued RHP describing the temperature correlation function of local fields (1.1) was formulated in [3]. We have presented already the general form of this RHP in the previous section. Here we give the explicit expression for the jump matrix and explain all necessary definitions and notations.

According to [3], the operator-valued RHP in question consists of finding the operator $\hat{\chi}(\lambda)$ possessing the properties,

$$\begin{aligned}
&1^a.\ \hat{\chi}(\lambda) \to \hat{I}, \quad \lambda \to \infty,\\
&2^a.\ \hat{\chi}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R,\\
&3^a.\ \hat{\chi}_-(\lambda) = \hat{\chi}_+(\lambda)\hat{G}(\lambda), \qquad \lambda \in R,
\end{aligned} \tag{3.1}$$

where the operator-valued entries of the jump matrix $\hat{G}(\lambda)$ are defined by the equations,

$$\begin{aligned}
\hat{G}_{11}(\lambda|u,v) &= \delta(u-v) - Z(v,\lambda)\delta(u-\lambda)\vartheta(\lambda)e^{\phi_D(\lambda)};\\
\hat{G}_{12}(\lambda|u,v) &= 2\pi i(\vartheta(\lambda)-1)\delta(u-\lambda)\delta(v-\lambda)e^{\psi(\lambda)+\tau(\lambda)};\\
\hat{G}_{21}(\lambda|u,v) &= -\frac{i}{2\pi}Z(u,\lambda)Z(v,\lambda)\vartheta(\lambda)e^{\phi_A(\lambda)+\phi_D(\lambda)-\psi(\lambda)-\tau(\lambda)};\\
\hat{G}_{22}(\lambda|u,v) &= \delta(u-v) - Z(u,\lambda)\delta(v-\lambda)\vartheta(\lambda)e^{\phi_A(\lambda)}.
\end{aligned} \tag{3.2}$$

Here $\phi_A(\lambda)$, $\phi_D(\lambda)$ and $\psi(\lambda)$ are the dual fields (2.22). Recall that starting from this section we consider $\exp\phi_A(\lambda)$, $\exp\phi_D(\lambda)$, and $\exp\psi(\lambda)$ as the classical functions which are rational and bounded in a finite strip ($|\operatorname{Im}\lambda| < c/2$) near the real axis. Moreover, we will assume (see the arguments given in the end of this section) the following symmetric properties of the functions $\psi(\lambda)$ and $\phi(\lambda) \equiv \phi_A(\lambda) - \phi_D(\lambda)$:

$$\psi(\lambda) = \bar{\psi}(\bar{\lambda}), \tag{3.3}$$

and

$$\phi(\lambda) = -\bar{\phi}(\bar{\lambda}), \tag{3.4}$$

The function $Z(\lambda,\mu)$ is equal to

$$Z(\lambda,\mu) = \frac{e^{-\phi_D(\lambda)}}{h(\mu,\lambda)} + \frac{e^{-\phi_A(\lambda)}}{h(\lambda,\mu)}, \tag{3.5}$$

where $h(\lambda,\mu) = (\lambda-\mu+ic)/ic$ was introduced in (2.23). The Fermi weight $\vartheta(\lambda)$ was defined in (1.8), and it is rapidly decreasing function at $\lambda\to\infty$. The only object depending on the time t and the distance x is the function $\tau(\lambda)\equiv\tau(\lambda;x,t)$ defined by the equation :

$$\tau(\lambda) = it\lambda^2 - ix\lambda, \tag{3.6}$$

and representing the dispersion law of the underlying classical model.

In the present paper we study the large time and the long distance asymptotic behavior of the solution of the RHP (a). More precisely we consider the case when $t\to\infty$, $x\to\infty$, while their ratio remains fixed:

$$x/2t \equiv \lambda_0 = O(1).$$

Therefore it is convenient to use new independent variables t and λ_0 instead of t and x. The function $\tau(\lambda)$ then turns into

$$\tau(\lambda) = it(\lambda-\lambda_0)^2 - it\lambda_0^2. \tag{3.7}$$

Since the jump matrix $\hat{G}(\lambda)$ (3.2) depends on delta-functions whose arguments contain parameter λ, one has to understand the jump condition 3^a in a weak sense. In order to avoid the

complications caused by the presence of the distributions depending on the complex parameter, we make the following regularization of the jump matrix (3.2) .

Let

$$\begin{aligned} \delta_\epsilon(u-\lambda) &= \frac{1}{2\sqrt{\pi\epsilon}} e^{-\frac{(u-\lambda)^2}{4\epsilon}}, \\ \mathcal{N}_\epsilon(\lambda) &= \int_{-\infty}^{\infty} Z^2(u,\lambda)\delta_\epsilon(u-\lambda)\,du, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} |1\rangle \equiv |1,u\rangle &= \frac{\delta_\epsilon(u-\lambda)Z(u,\lambda)}{\sqrt{\mathcal{N}_\epsilon(\lambda)Z(\lambda,\lambda)}}, \quad \langle 1| \equiv \langle 1,v| = \sqrt{\frac{Z(\lambda,\lambda)}{\mathcal{N}_\epsilon(\lambda)}}Z(v,\lambda), \\ |2\rangle \equiv |2,u\rangle &= \sqrt{\frac{Z(\lambda,\lambda)}{\mathcal{N}_\epsilon(\lambda)}}Z(u,\lambda), \quad \langle 2| \equiv \langle 2,v| = \frac{\delta_\epsilon(v-\lambda)Z(v,\lambda)}{\sqrt{\mathcal{N}_\epsilon(\lambda)Z(\lambda,\lambda)}}. \end{aligned} \tag{3.9}$$

Obviously

$$\langle 1|1\rangle \equiv \int_{-\infty}^{\infty}\langle 1,u|1,u\rangle\,du = \langle 2|2\rangle \equiv \int_{-\infty}^{\infty}\langle 2,u|2,u\rangle\,du = 1. \tag{3.10}$$

Also, in virtue of the symmetry equation (3.4), one can see that $\mathcal{N}_\epsilon(\lambda)$ has no zeros on the real line.

We define the entries of the regularized jump matrix by the equations,

$$\begin{aligned} \hat{G}_{11}(\lambda) &= \hat{\imath} - \vartheta(\lambda)Z(\lambda,\lambda)e^{\phi_D(\lambda)}|1\rangle\langle 1|; \\ \hat{G}_{12}(\lambda) &= 2\pi i(\vartheta(\lambda)-1)Z(\lambda,\lambda)e^{\psi(\lambda)+\tau(\lambda)}|1\rangle\langle 2|; \\ \hat{G}_{21}(\lambda) &= -\frac{i}{2\pi}\vartheta(\lambda)Z(\lambda,\lambda)e^{\phi_A(\lambda)+\phi_D(\lambda)-\psi(\lambda)-\tau(\lambda)}|2\rangle\langle 1|; \\ \hat{G}_{22}(\lambda) &= \hat{\imath} - \vartheta(\lambda)Z(\lambda,\lambda)e^{\phi_A(\lambda)}|2\rangle\langle 2|, \end{aligned} \tag{3.11}$$

where $\hat{\imath} \equiv \delta(u-v)$. It is easy to see that in the limit $\epsilon \to 0$ the jump matrix (3.11) turns into (3.2).

The RHP (a) with the regularized jump matrix corresponds to a new regularized operator $\mathcal{B}$. In particular, the Fredholm determinant $\det\left(\tilde{I}+\widetilde{V}\right)$ should be replaced by $\det\left(\tilde{I}+\widetilde{V}_\epsilon\right)$ (see appendix B). New regularized kernel $\widetilde{V}_\epsilon$, as well as original kernel $\widetilde{V}$, is not singular, and the limit $\widetilde{V}_\epsilon \to \widetilde{V}, \quad \epsilon \to 0$ is well defined. It is remarkable that all relationships (2.19)–(2.21) remain valid for the regularized objects for *finite* values of ϵ (not only in the limit $\epsilon \to 0$!). In the rest of the paper we will only work with the regularized operator $\widetilde{V}_\epsilon$, and we will denote it as $\widetilde{V}$ omitting the subscript ϵ. We will also assume that the large time limit we study commutes with the limit $\epsilon \to 0$. As an indirect justification of this conjecture, we take another special property of our regularization which will be revealed later: the first leading terms which we evalute for the regularized Fredholm determinant *do not depend on* ϵ.

As in [3], under the assumptions made in the previous section concerning the general theory of the operator-valued RHPs we are dealing with, solvability of the RHP (3.1, 3.11) implies uniqness of its solution. Moreover, using the representation property (1.20) and equation (2.14) we conclude that the solution $\hat{\chi}(\lambda)$ of the RHP (a) with the jump matrix (3.11) satisfies the equation,

$$\det\hat{\chi}(\lambda)=1\quad\text{for all}\quad\lambda\,. \tag{3.12}$$

The very possibility of solving the RHP (3.1, 3.11) asymptoticly as $t\to\infty$ is based on the presence of the rapidly oscillating exponents $e^{\pm\tau(\lambda)}$ in the jump matrix. In the next section we start the asymptotic analysis of the solution of the RHP (3.1, 3.11) using an operator-valued generalization of the nonlinear steepest descend method suggested in [38] for the oscillatory matrix-valued Riemann-Hilbert problems.

We conclude this section by one important note.

The model QNLS has two different phases, corresponding to the positive and negative chemical potential h in the Hamiltonian (1.5). In particular, the ground state of the model for $h<0$ coincides with the bare Fock vacuum, while for $h>0$ the ground state is the Dirac sea [48]. At finite temperature the difference is not so essential, but nevertheless one could expect that the asymptotics of the correlation function is different for these two cases. Since this asymptotics is defined by the solution of the RHP (a), the properties of the jump matrix (3.11) should be different for $h>0$ and $h<0$. This fact does take place in the free fermionic limit [29] and it is related to the zeros of the diagonal elements of the jump matrix. Out off free fermionic point one should consider zeros of the determinants of $\hat{G}_{11}$ and $\hat{G}_{22}$. Since both of these operators are the sum of identity operator and one-dimensional projector, it is easy to see that

$$\begin{aligned}\det\hat{G}_{11}(\lambda)&=1-\vartheta(\lambda)Z(\lambda,\lambda)e^{\phi_D(\lambda)}=\frac{e^{\frac{\varepsilon(\lambda)}{T}}-e^{-\phi(\lambda)}}{e^{\frac{\varepsilon(\lambda)}{T}}+1},\\ \det\hat{G}_{22}(\lambda)&=1-\vartheta(\lambda)Z(\lambda,\lambda)e^{\phi_A(\lambda)}=\frac{e^{\frac{\varepsilon(\lambda)}{T}}-e^{\phi(\lambda)}}{e^{\frac{\varepsilon(\lambda)}{T}}+1},\end{aligned} \tag{3.13}$$

In the free fermionic point (coupling constant c goes to infinity) one can put dual fields in these formulæ equal to zero, since due to commutation relations (2.23), the vacuum expectation value is trivial in this limit. As we had mentioned in the Introduction, the solution of the Yang–Yang equation (1.6) $\varepsilon(\lambda)$ is positive, if $h<0$, and $\varepsilon(\lambda)$ has two real roots, if $h>0$. Therefore the determinants of $\hat{G}_{11}$ and $\hat{G}_{22}$ possess just the same properties.

Out off free fermionic point, in order to answer the question, whether the determinants of $\hat{G}_{11}$ and $\hat{G}_{22}$ have zeros at the real axis, one need to know the properties of the function $\exp\{\varepsilon(\lambda)/T\}-\exp\{\pm\phi(\lambda)\}$. These properties must reflect by a natural way the properties of the operator $\exp\{\varepsilon(\lambda)/T\}-\exp\{\pm\phi(\lambda)\}$. It was shown in [1] that dual fields can be expressed in terms of

canonical Bose fields. Using these representations one can prove, in particular, that $\psi^\dagger(\bar\lambda)=\psi(\lambda)$ and $\phi^\dagger(\bar\lambda)=-\phi(\lambda)$. Therefore we demand the corresponding classical functions to possess similar properties (3.3), (3.4). The operator $\exp\{\pm\phi(\lambda)\}$ obviously is a unitary operator with the spectrum, belonging to the unite circle. There are serious reasons to assume that all the points of the unite circle belong to the spectrum of this operator (although this fact have not been checked rigorously). Therefore, if $h<0$, and hence $\varepsilon(\lambda)>0$, we conclude that the spectrum of the operator $\exp\{\varepsilon(\lambda)/T\}-\exp\{\pm\phi_A(\lambda)\}$ does not contain zero:

$$0\notin\sigma\left(e^{\varepsilon(\lambda)/T}-e^{\pm\phi(\lambda)}\right),\qquad h<0. \tag{3.14}$$

On the contrary for $h>0$ the function $\varepsilon(\lambda)$ has two real roots, hence

$$0\in\sigma\left(e^{\varepsilon(\lambda)/T}-e^{\pm\phi(\lambda)}\right),\qquad h>0. \tag{3.15}$$

Thus, taking into account (3.14), (3.15), we shall demand the classical function $\phi(\lambda)=\phi_A(\lambda)-\phi_D(\lambda)$ to be such, that the determinants of the operators $\hat G_{11}$ and $\hat G_{22}$ have no real roots for $h<0$, and they both have two real roots for $h>0$.

4 Negative chemical potential

In this section we begin the asymptotic analysis of the RHP (a) with the jump matrix (3.11). We refer the reader to the works [39] and [37], which present the nonlinear steepest descent method for the classical NLS, to see that each of our basic operator constructions has its matrix counterpart. In particular, in this and the next sections we perform the first step of the method, i.e. the transformation of the jump matrix to the proper up-and-low-triangular forms followed by the deformation of the real line to the steepest descent contours with respect of the exponent $\tau(\lambda)$. All the considerations of this section are very close to those of [45].

We start with the case of negative chemical potential. Let as before λ_0 denote the saddle point of the exponent $\tau(\lambda)$, i.e. (see (3.6) - (3.7)),

$$\lambda_0=\frac{x}{2t}\,, \tag{4.1}$$

and consider the following substitution

$$\hat\chi(\lambda)=\hat\Phi(\lambda)\hat{\mathcal R}(\lambda), \tag{4.2}$$

where $\hat\Phi(\lambda)$ is new unknown operator-valued matrix, and $\hat{\mathcal R}(\lambda)$ is diagonal matrix

$$\hat{\mathcal R}=\left(\begin{array}{cc}\hat\varrho(\lambda) & 0\\ 0 & \left(\hat\varrho^T(\lambda)\right)^{-1}\end{array}\right). \tag{4.3}$$

Here $\hat{\varrho}(\lambda)$ is the solution of the RHP

$$
\begin{aligned}
&1^b.\ \hat{\varrho}(\lambda) \to \hat{\imath} \equiv \delta(u-v), \quad \lambda \to \infty,\\
&2^b.\ \hat{\varrho}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R,\\
&3^b.\ \hat{\varrho}_-(\lambda) = \hat{\varrho}_+(\lambda)\left(\theta(\lambda_0-\lambda)\hat{G}_{11} + \theta(\lambda-\lambda_0)(\hat{G}^T_{22})^{-1}\right), \quad \lambda \in R,
\end{aligned}
\tag{4.4}
$$

and its kernel $\hat{\varrho}(\lambda|u,v)$ is integrable at the point $\lambda = \lambda_0$ (cf. the function $\delta(\lambda)$ in [39], [37]). The RHP (b) looks like a scalar RHP, however one should remember that $\hat{\varrho}(\lambda) \equiv \hat{\varrho}(\lambda|u,v)$ and $\hat{G}_{jk}(\lambda) \equiv \hat{G}_{jk}(\lambda|u,v)$ are integral operators. The symbols $\hat{\varrho}^T(\lambda)$ as well as $\hat{G}^T_{22}(\lambda)$ mean the transposition of these operators: $\hat{\varrho}^T(\lambda|u,v) = \hat{\varrho}(\lambda|v,u)$ and $\hat{G}^T_{22}(\lambda|u,v) = \hat{G}_{22}(\lambda|v,u)$ respectively. $\theta(\lambda)$ is the step function

$$
\theta(\lambda) = \begin{cases} 1, & \lambda > 0,\\ 0, & \lambda < 0. \end{cases}
\tag{4.5}
$$

Since the jump operator in (4.4) is discontinuous at $\lambda = \lambda_0$ we have demanded an additional condition of integrability for the solution of the RHP (b) (cf. the standard conditions [40] of the theory of factorization of matrix functions). Recall also, that we consider the case, when $\det \hat{G}_{11}$ and $\det \hat{G}_{22}$ are not equal to zero at the real axis, hence the jump operator $\theta(\lambda_0-\lambda)\hat{G}_{11}(\lambda)+\theta(\lambda-\lambda_0)\left(\hat{G}^T_{22}(\lambda)\right)^{-1}$ is well defined.

Thus we obtain a new RHP for $\hat{\Phi}(\lambda)$

$$
\begin{aligned}
&1^c.\ \hat{\Phi}(\lambda) \to \hat{I}, \quad \lambda \to \infty,\\
&2^c.\ \hat{\Phi}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R,\\
&3^c.\ \hat{\Phi}_-(\lambda) = \hat{\Phi}_+(\lambda)\hat{G}_\Phi(\lambda), \qquad \lambda \in R,
\end{aligned}
\tag{4.6}
$$

where

$$
\hat{G}_\Phi(\lambda) = \hat{\mathcal{R}}_+(\lambda)\hat{G}(\lambda)\hat{\mathcal{R}}_-^{-1}(\lambda).
\tag{4.7}
$$

The coefficients $\hat{b}$ and $\hat{c}$ of the asymptotic expansion of the original operator $\hat{\chi}(\lambda)$ (the RHP (a)) can be easily expressed in terms of asymptotic expansions of $\hat{\Phi}(\lambda)$ and $\hat{\varrho}(\lambda)$:

$$
\hat{\Phi} = \hat{I} + \frac{\hat{\Phi}_0}{\lambda} + \frac{\hat{\Phi}_1}{\lambda^2} + \dots,
\tag{4.8}
$$

$$
\hat{\varrho} = \hat{\imath} + \frac{\hat{\varrho}_0}{\lambda} + \frac{\hat{\varrho}_1}{\lambda^2} + \dots.
\tag{4.9}
$$

In particular, it is easy to see that

$$
\begin{aligned}
&\hat{b}_{11} = (\hat{\Phi}_0)_{11} + \hat{\varrho}_0,\\
&\hat{c}_{22} - \hat{c}_{11} = (\hat{\Phi}_1)_{22} - (\hat{\Phi}_1)_{11} - (\hat{\Phi}_0)_{22}\hat{\varrho}^T_0 - (\hat{\Phi}_0)_{11}\hat{\varrho}_0 - \hat{\varrho}_1 - \hat{\varrho}^T_1 + \left(\hat{\varrho}^T_0\right)^2.
\end{aligned}
\tag{4.10}
$$

Consider the function $\Delta(\lambda) = \det \hat{\varrho}(\lambda)$. This is not an operator but a complex function. The asymptotic expansion of this function at $\lambda \to \infty$ plays an important role. It is convinient to write down this expansion in the exponential form,

$$\Delta = \exp\left\{\frac{\Delta_0}{\lambda} + \frac{\Delta_1}{\lambda^2} + \dots\right\}. \tag{4.11}$$

In order to find the first logarithmic derivatives of the Fredholm determinant we need to know only the traces of the operators $\hat{b}_{11}$ and $\hat{c}_{22} - \hat{c}_{11}$ (see (2.19)). Therefore taking the trace of (4.10) we arrive at

$$\begin{aligned} \operatorname{tr} \hat{b}_{11} &= \operatorname{tr}(\hat{\Phi}_0)_{11} + \Delta_0, \\ \operatorname{tr}(\hat{c}_{22} - \hat{c}_{11}) &= \operatorname{tr}\left[(\hat{\Phi}_1)_{22} - (\hat{\Phi}_1)_{11} - (\hat{\Phi}_0)_{22}\hat{\varrho}_0^T - (\hat{\Phi}_0)_{11}\hat{\varrho}_0\right] - 2\Delta_1. \end{aligned} \tag{4.12}$$

Consider now the jump matrix $\hat{G}_\Phi$. Using the equations,

$$(|1\rangle\langle 1|)^T = |2\rangle\langle 2|, \qquad (|k\rangle\langle j|)^T = |k\rangle\langle j|, \quad k \neq j, \tag{4.13}$$

and

$$(|k\rangle\langle j|)(|j\rangle\langle l|) = |k\rangle\langle l|, \quad k, j, l = 1, 2, \tag{4.14}$$

one can easily check that

$$\begin{aligned} \hat{G}_{22} - \hat{G}_{21}(\hat{G}_{11})^{-1}\hat{G}_{12} &= (\hat{G}_{11}^T)^{-1}, \\ \hat{G}_{11} - \hat{G}_{12}(\hat{G}_{22})^{-1}\hat{G}_{21} &= (\hat{G}_{22}^T)^{-1}. \end{aligned} \tag{4.15}$$

Due to these properties the jump matrix $\hat{G}_\Phi$ has the form

$$\hat{G}_\Phi = \begin{pmatrix} \hat{\imath} & \hat{P}e^{\tau} \\ \hat{Q}e^{-\tau} & \hat{\imath} + \hat{Q}\hat{P} \end{pmatrix}, \qquad \text{for} \quad \lambda < \lambda_0 \tag{4.16}$$

where

$$\hat{Q}e^{-\tau} = (\hat{\varrho}_+^T)^{-1}\hat{G}_{21}(\hat{G}_{11})^{-1}\hat{\varrho}_+^{-1}, \tag{4.17}$$

$$\hat{P}e^{\tau} = \hat{\varrho}_-(\hat{G}_{11})^{-1}\hat{G}_{12}\hat{\varrho}_-^T, \tag{4.18}$$

and

$$\hat{G}_\Phi = \begin{pmatrix} \hat{\imath} + \hat{\tilde{P}}\hat{\tilde{Q}} & \hat{\tilde{P}}e^{\tau} \\ \hat{\tilde{Q}}e^{-\tau} & \hat{\imath} \end{pmatrix}, \qquad \text{for} \quad \lambda > \lambda_0 \tag{4.19}$$

where

$$\hat{\tilde{Q}}e^{-\tau} = (\hat{\varrho}_-^T)^{-1}(\hat{G}_{22})^{-1}\hat{G}_{21}\hat{\varrho}_-^{-1}, \tag{4.20}$$

$$\hat{\tilde{P}}e^{\tau} = \hat{\varrho}_+\tilde{G}_{12}(\tilde{G}_{22})^{-1}\hat{\varrho}_+^T. \tag{4.21}$$

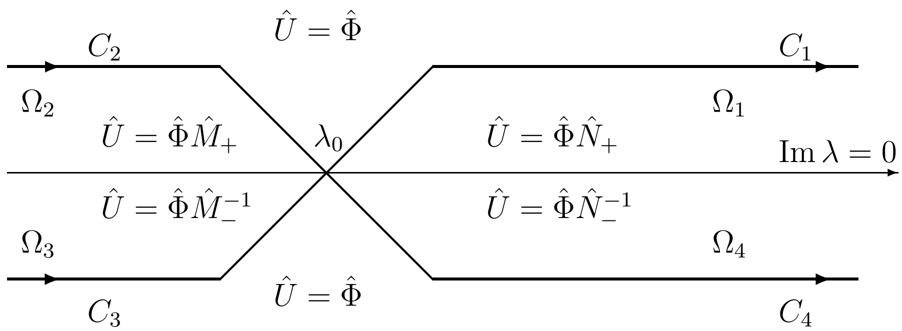

Figure 1: Contour for new RH problem

Therefore one can factorize the jump matrix

$$\begin{aligned} \hat{G}_\Phi &= \hat{M}_+\hat{M}_-, \qquad \text{for} \quad \lambda < \lambda_0, \\ \hat{G}_\Phi &= \hat{N}_+\hat{N}_-, \qquad \text{for} \quad \lambda > \lambda_0, \end{aligned} \tag{4.22}$$

where

$$\hat{M}_+ = \begin{pmatrix} \hat{\imath} & 0 \\ \hat{Q}e^{-\tau} & \hat{\imath} \end{pmatrix}, \qquad \hat{M}_- = \begin{pmatrix} \hat{\imath} & \hat{P}e^{\tau} \\ 0 & \hat{\imath} \end{pmatrix}, \tag{4.23}$$

$$\hat{N}_+ = \begin{pmatrix} \hat{\imath} & \hat{\hat{P}}e^{\tau} \\ 0 & \hat{\imath} \end{pmatrix}, \qquad \hat{N}_- = \begin{pmatrix} \hat{\imath} & 0 \\ \hat{\hat{Q}}e^{-\tau} & \hat{\imath} \end{pmatrix}. \tag{4.24}$$

Notice that $\hat{Q}$ and $\hat{\hat{P}}$ can be analytically continued into a neighborhood of the real axis in the upper half-plane. Similarly, $\hat{\hat{Q}}$ and $\hat{P}$ can be continued into a strip in the lower half-plane.

Consider new contour, which is shown on the Fig.1 The contour consists of four branches C_1, C_2, C_3 and C_4, having the origin in the saddle point λ_0 (arrows show the positive direction). The branches C_j and the real axis are the boundaries of the four half-infinite 'wedges' Ω_j, $(j = 1, \dots, 4)$. Matrices $N_\pm$ can be analytically continued into the wedges Ω_1 and Ω_4 respectively, matrices $M_\pm$ — into the wedges Ω_2 and Ω_3.

Define the matrix $\hat{U}$ as follows:

$$\begin{aligned} \hat{U} &= \hat{\Phi}, \qquad \lambda \notin \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \\ \hat{U} &= \hat{\Phi}\hat{N}_+, \qquad \lambda \in \Omega_1, \\ \hat{U} &= \hat{\Phi}\hat{M}_+, \qquad \lambda \in \Omega_2, \\ \hat{U} &= \hat{\Phi}\hat{M}_-^{-1}, \qquad \lambda \in \Omega_3, \\ \hat{U} &= \hat{\Phi}\hat{N}_-^{-1}, \qquad \lambda \in \Omega_4, \end{aligned} \tag{4.25}$$

(see Fig.1). Then the matrix $\hat{U}$ has no jump at the real axis, however it has jumps at the new contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$. The RHP for $\hat{U}$ can be formulated now as

$$1^d.\ \hat{U}(\lambda) \to \hat{I} \quad \lambda \to \infty,$$

$$2^d.\ \hat{U}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin C,$$

$$3^d.\ \hat{U}_- = \hat{U}_+ \hat{N}_+, \qquad \lambda \in C_1, \tag{4.26}$$

$$\hat{U}_- = \hat{U}_+ \hat{M}_+, \qquad \lambda \in C_2, \tag{4.27}$$

$$\hat{U}_- = \hat{U}_+ \hat{M}_-, \qquad \lambda \in C_3, \tag{4.28}$$

$$\hat{U}_- = \hat{U}_+ \hat{N}_-, \qquad \lambda \in C_4. \tag{4.29}$$

where $\hat{U}_\mp$ are boundary values of $\hat{U}$ from the right and left of the contour C respectively.

It is easy to see that all jump matrices in (4.26)–(4.29) have the form

$$\hat{I} + o(e^{-t^{2\delta}}), \qquad \text{for} \quad |\lambda - \lambda_0| > t^{-1/2+\delta}, \tag{4.30}$$

where δ is an arbitrary small positive number. Thus, we can put $\hat{N}_\pm = \hat{M}_\pm \approx \hat{I}$ up to exponentially decreasing corrections for all $\lambda \in C$ except small vicinity of the saddle point. Formally appealing to the integral equation (1.15), written for the contour C and the operator-valued function $\hat{U}_+$ (cf. the *nonformal* arguments of [39] for the usual matrix case), we conclude that

$$\hat{U} \approx \hat{I}, \qquad \Rightarrow \qquad \hat{\Phi} \approx \hat{I}, \qquad \Rightarrow \qquad \hat{\chi} \approx \hat{\mathcal{R}}. \tag{4.31}$$

and hence the coefficients of the asymptotic expansion $\hat{\Phi}_0$ and $\hat{\Phi}_1$ are asymptoticly equal to zero:

$$\hat{\Phi}_0,\ \hat{\Phi}_1 \approx 0. \tag{4.32}$$

Thus, similar to [45], the traces of the operators $\hat{b}_{11}$ and $\hat{c}_{22} - \hat{c}_{11}$ asymptoticly depend only on the coefficients of the λ - expansion (4.11) of the function $\Delta(\lambda)$ (see (4.12)):

$$\begin{aligned} &\operatorname{tr} \hat{b}_{11} = \Delta_0 + o(1), \\ &\operatorname{tr}(\hat{c}_{22} - \hat{c}_{11}) = -2\Delta_1 + o(1). \end{aligned} \tag{4.33}$$

On the other hand, one can easily find $\Delta(\lambda)$ since the RHP (b) for the operator $\hat{\varrho}(\lambda)$ implies the following scalar RHP for its determinant $\Delta(\lambda)$ (cf. (2.11) - (2.13)):

$$\begin{aligned} &1^e.\ \Delta(\lambda) \to 1, \quad \lambda \to \infty, \\ &2^e.\ \Delta(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R, \\ &3^e.\ \Delta_-(\lambda) = \Delta_+(\lambda) \det\Big(\theta(\lambda_0 - \lambda)\hat{G}_{11} + \theta(\lambda - \lambda_0)(\hat{G}_{22}^T)^{-1}\Big), \quad \lambda \in R, \end{aligned} \tag{4.34}$$

Using the fact that both of the operators $\hat{G}_{11}$ and $\hat{G}_{22}$ are a sum of the identity operator $\hat{\imath}$ and an one-dimensional projector we find

$$(\hat{G}_{22}^T)^{-1} = \hat{\imath} + \frac{Z(\lambda,\lambda)\vartheta(\lambda)e^{\phi_A(\lambda)}}{1 - Z(\lambda,\lambda)\vartheta(\lambda)e^{\phi_A(\lambda)}}|1\rangle\langle 1|, \tag{4.35}$$

and (cf. (1.20))

$$\det\left(\theta(\lambda_0-\lambda)\hat{G}_{11}(\lambda) + \theta(\lambda-\lambda_0)\Big(\hat{G}_{22}^T(\lambda)\Big)^{-1}\right) = \Big(1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\lambda_0)}\Big)\Big)^{\mathrm{sign}(\lambda_0-\lambda)}, \tag{4.36}$$

where $\mathrm{sign}(\lambda) = \theta(\lambda) - \theta(-\lambda)$ is the sign function, and $\phi(\lambda) = \phi_A(\lambda) - \phi_D(\lambda)$. Thus from (2.14) it follows that the solution of the RHP (e) is given by the explicit formula

$$\Delta = \exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\,\mathrm{sign}(\lambda_0-\mu)\ln\Big(1-\vartheta(\mu)\Big(1+e^{\phi(\mu)\,\mathrm{sign}(\mu-\lambda_0)}\Big)\Big)\right\}, \tag{4.37}$$

and, hence,

$$\Delta_0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu\,\mathrm{sign}(\lambda_0-\mu)\ln\Big(1-\vartheta(\mu)\Big(1+e^{\phi(\mu)\,\mathrm{sign}(\mu-\lambda_0)}\Big)\Big), \tag{4.38}$$

$$\Delta_1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mu d\mu\,\mathrm{sign}(\lambda_0-\mu)\ln\Big(1-\vartheta(\mu)\Big(1+e^{\phi(\mu)\,\mathrm{sign}(\mu-\lambda_0)}\Big)\Big). \tag{4.39}$$

The logarithmic derivatives of the Fredholm determinant with respect to the variables t and λ_0 are equal to

$$\begin{aligned}&\frac{1}{2t}\partial_{\lambda_0}\ln\det\Big(\tilde{I}+\widetilde{V}\Big) = i\,\mathrm{tr}\,\hat{b}_{11} = i\Delta_0 + o(1),\\&\left(\partial_t - \frac{\lambda_0}{t}\partial_{\lambda_0}\right)\ln\det\Big(\tilde{I}+\widetilde{V}\Big) = i\,\mathrm{tr}(\hat{c}_{22}-\hat{c}_{11}) = -2i\Delta_1 + o(1).\end{aligned} \tag{4.40}$$

Integrating these equations with respect to t and λ_0 we arrive at

$$\ln\det(\tilde{I}+\tilde{V}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu|x-2\mu t|\ln\Big(1-\vartheta(\mu)\Big(1+e^{\phi(\mu)\,\mathrm{sign}(\mu-\lambda_0)}\Big)\Big) + \mathcal{O}(\ln t), \tag{4.41}$$

where $x = 2t\lambda_0$, $\quad t\to\infty$. It is worth observing that the leading term in the r.h.s. of (4.41) does not depend on the regularization parameter ϵ. It is also should be mentioned that equation (4.41) for the asymptotics of the Fredholm determinant exactly coincides with the one obtained in [4] by independent and more direct methods. This fact partially justifies our, mostly formal, manipulations with the operator-valued Riemann-Hilbert problems.

5 Positive chemical potential

In the case of positive chemical potential the method of factorization, considered in the section 4 is not applicable directly to the jump matrix (3.11). Indeed, now the determinants of the

operators $\hat{G}_{11}(\lambda)$ and $\hat{G}_{22}(\lambda)$ have zeros on the real axis, and the scalar RHP (e) for the function $\Delta(\lambda) = \det \hat{\varrho}(\lambda)$ is ill posed. The factorization (4.22) also does not exist, since the operators $\hat{Q}$, $\hat{P}$, $\hat{\tilde{P}}$ and $\hat{\tilde{Q}}$ have singularities on the real axis (see (4.17) - (4.21)).

In this situation one can use the approach similar to the one proposed for the free fermionic limit of the problem in [29]. First, we have to specify the position of the determinants roots relative to the saddle point λ_0. Denote the roots of the function,

$$\det \hat{G}_{11}(\lambda) = 1 - \vartheta(\lambda) Z(\lambda,\lambda) e^{\phi_D(\lambda)} = 1 - \vartheta(\lambda)(1 + e^{-\phi(\lambda)}),$$

$$\phi(\lambda) = \phi_A(\lambda) - \phi_D(\lambda),$$

as Λ_j, $j = 1, 2$, $\Lambda_1 < \Lambda_2$. We shall concentrate our attention on the case,

$$\Lambda_1 < \Lambda_2 < \lambda_0,$$

assuming simultaniously that the roots of $\det \hat{G}_{22}(\lambda)$ lie to the left of λ_0 as well (see the arguments in the end of the section 3). All the other possible cases of the roots position can be considered in a similar fashion. Moreover, similar to the free fermion case [29], the difference in the way how the roots are distributed along the real line does not affect the final answer for the asymptotics. What is important is the assumption that non of the roots is close to the saddle point.

Secondly, observe that the integral operator $\tilde{I} + \widetilde{V}$ can be continued into some vicinity of the real axis, hence the integration contour can be slightly deformed in this vicinity. The Fredholm determinant evidently does not depend on such deformation.

The deformed contour Γ is shown on the Fig.2. We will see below that this particular choice

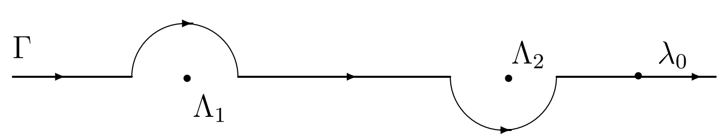

Figure 2: Deformation of the contour

of the contour deformation will guarantee the solvability of the deformed scalar problem (4.34) for the function $\Delta(\lambda)$.

As in the case of negative chemical potential we make the substitution

$$\hat{\chi}(\lambda) = \hat{\Phi}(\lambda)\hat{\mathcal{R}}(\lambda), \tag{5.1}$$

where as before $\hat{\mathcal{R}}(\lambda)$ is diagonal matrix

$$\hat{\mathcal{R}} = \begin{pmatrix} \hat{\varrho}(\lambda) & 0 \\ 0 & (\hat{\varrho}^T(\lambda))^{-1} \end{pmatrix}, \tag{5.2}$$

and $\hat{\varrho}(\lambda)$ satisfies the RHP

$$\begin{aligned} &1^f.\ \hat{\varrho}(\lambda) \to \hat{\imath}, \quad \lambda \to \infty, \\ &2^f.\ \hat{\varrho}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin \Gamma, \\ &3^f.\ \hat{\varrho}_-(\lambda) = \hat{\varrho}_+(\lambda)\left(\theta(\lambda_0-\lambda)\hat{G}_{11} + \theta(\lambda-\lambda_0)(\hat{G}_{22}^T)^{-1}\right), \quad \lambda \in \Gamma, \end{aligned} \tag{5.3}$$

The difference between the Riemann–Hilbert problem (b), considered in the previous section, and RHP (f) is that the last one is formulated on the jump contour Γ instead of the real axis. Similary, both the operator-valued RHP for $\hat{\Phi}(\lambda)$ and the scalar RHP for $\Delta(\lambda) = \det \hat{\varrho}(\lambda)$ are set now on the contour Γ. Due to the choice made for the contour Γ, the function

$$\arg\det\left(\theta(\lambda_0-\lambda)\hat{G}_{11}(\lambda) + \theta(\lambda-\lambda_0)(\hat{G}_{22}^T)^{-1}(\lambda)\right)$$

can be defined in such way that it is continuous for all $\lambda \neq \lambda_0$ and approaches 0 as $\lambda \to \pm\infty$. In other words, the Δ-RHP has no index and hence is well posed. Using again the general formula (2.14) (with the real line replaced by Γ), we have that

$$\Delta = \exp\left\{-\frac{1}{2\pi i}\int_\Gamma \frac{d\mu}{\mu-\lambda}\,\mathrm{sign}(\lambda_0 - \Re\mu)\ln\left(1-\vartheta(\mu)\left(1+e^{\phi(\mu)\,\mathrm{sign}(\Re\mu-\lambda_0)}\right)\right)\right\}, \tag{5.4}$$

(cf. (4.37)).

The jump matrix $\hat{G}_\Phi$ is given by the same equations (4.16)- (4.21) as before. The operators $\hat{Q}$ and $\hat{P}$ have singularities at $\Lambda_{1,2}$, and hence the obstructions to the analytic continuation of the operator matrices $\hat{M}_+$ and $\hat{M}_-$ occure (the singularities of $\hat{N}_\pm$ are not relevant since we have assumed that the zeros of $\det \hat{G}_{22}$ lie to the left of λ_0). Therefore, before factorizing the $\hat{\Phi}$- jump matrix let us make one more substitution (cf. [29]):

$$\hat{\Phi}(\lambda) = (\lambda\hat{I} + \hat{\Pi})\hat{\Phi}^{(0)}(\lambda)\begin{pmatrix} (\lambda-\Lambda_2)\hat{\imath} & 0 \\ 0 & (\lambda-\Lambda_1)\hat{\imath} \end{pmatrix}^{-1}. \tag{5.5}$$

Here $\hat{\Pi}$ is an operator satisfying the following conditions:

$$\begin{aligned} (\Lambda_2\hat{I} + \hat{\Pi})\hat{\Phi}^{(0)}(\Lambda_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\ (\Lambda_1\hat{I} + \hat{\Pi})\hat{\Phi}^{(0)}(\Lambda_1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \end{aligned} \tag{5.6}$$

The RHP for the operator $\hat{\Phi}^{(0)}(\lambda)$ has the form

$$
\begin{aligned}
&1^g.\ \hat{\Phi}^{(0)}(\lambda) \to \hat{I}, \quad \lambda \to \infty, \\
&2^g.\ \hat{\Phi}^{(0)}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin R, \\
&3^g.\ \hat{\Phi}^{(0)}_{-}(\lambda) = \hat{\Phi}^{(0)}_{+}(\lambda)\hat{G}_s(\lambda), \qquad \lambda \in R,
\end{aligned}
\tag{5.7}
$$

where the jump matrices $\hat{G}_s$ and $\hat{G}_\Phi$ (4.7) are related by

$$
\hat{G}_s(\lambda) = \begin{pmatrix} (\lambda-\Lambda_2)\hat{\imath} & 0 \\ 0 & (\lambda-\Lambda_1)\cdot\hat{\imath} \end{pmatrix}^{-1} \hat{G}_\Phi(\lambda) \begin{pmatrix} (\lambda-\Lambda_2)\hat{\imath} & 0 \\ 0 & (\lambda-\Lambda_1)\hat{\imath} \end{pmatrix}. \tag{5.8}
$$

Of course, now one should choose for $\hat{G}_\Phi$ the operator $\hat{\varrho}$ satisfying RHP (f).

Since the roots Λ_1 and Λ_2 as well as the roots of $\det \hat{G}_{22}(\lambda)$ all are smaller then λ_0, the factorization of the matrix $\hat{G}_s$ in the domain to the right from the saddle point produces the matrices $\hat{N}_\pm$ whose analyticity in the relevant half planes is obvious. In the domain $\lambda < \lambda_0$ the jump matrix $\hat{G}_s$ has the form

$$
\hat{G}_s(\lambda) = \begin{pmatrix} \hat{\imath} & \hat{P}^{(1)}e^{\tau} \\ \hat{Q}^{(1)}e^{-\tau} & \hat{\imath} + \hat{Q}^{(1)}\hat{P}^{(1)} \end{pmatrix}. \tag{5.9}
$$

Here

$$
\hat{P}^{(1)}(\lambda) = \hat{P}(\lambda)\left(\frac{\lambda-\Lambda_1}{\lambda-\Lambda_2}\right), \qquad \hat{Q}^{(1)}(\lambda) = \hat{Q}(\lambda)\left(\frac{\lambda-\Lambda_2}{\lambda-\Lambda_1}\right), \tag{5.10}
$$

and operators $\hat{Q}$ and $\hat{P}$ are given by (4.17) and (4.18). Thus, the jump matrix $\hat{G}_s$ can be factorized in the domain $\lambda < \lambda_0$ as

$$
\hat{G}_s = \hat{M}_+\hat{M}_-, \tag{5.11}
$$

where

$$
\hat{M}_+ = \begin{pmatrix} \hat{\imath} & 0 \\ \hat{Q}^{(1)}e^{-\tau} & \hat{\imath} \end{pmatrix}, \qquad \hat{M}_- = \begin{pmatrix} \hat{\imath} & \hat{P}^{(1)}e^{\tau} \\ 0 & \hat{\imath} \end{pmatrix}, \tag{5.12}
$$

Using explicit formulæ (3.11) for the operators $\hat{G}_{jk}$ one can write

$$
Q^{(1)}(\lambda) = q^{(1)}(\lambda)\left(\hat{\varrho}^T_+\right)^{-1}(\lambda)\,|2\rangle\langle 1|\,\hat{\varrho}^{-1}_+(\lambda), \tag{5.13}
$$

$$
P^{(1)}(\lambda) = p^{(1)}(\lambda)\hat{\varrho}_-(\lambda)\,|1\rangle\langle 2|\,\hat{\varrho}^T_-(\lambda), \tag{5.14}
$$

where

$$
p^{(1)}(\lambda) = \frac{2\pi i Z(\lambda,\lambda)(\vartheta(\lambda)-1)e^{\psi(\lambda)}}{1-Z(\lambda,\lambda)\vartheta(\lambda)e^{\phi_D(\lambda)}}\left(\frac{\lambda-\Lambda_1}{\lambda-\Lambda_2}\right), \tag{5.15}
$$

$$q^{(1)}(\lambda) = \frac{\vartheta(\lambda)Z(\lambda,\lambda)e^{\phi_A(\lambda)+\phi_D(\lambda)-\psi(\lambda)}}{2\pi i\left(1 - Z(\lambda,\lambda)\vartheta(\lambda)e^{\phi_D(\lambda)}\right)}\left(\frac{\lambda-\Lambda_2}{\lambda-\Lambda_1}\right). \tag{5.16}$$

Note that the denominator in these formulæ is just the determinant of the operator $\hat{G}_{11}$:

$$\det \hat{G}_{11}(\lambda) = 1 - Z(\lambda,\lambda)\vartheta(\lambda)e^{\phi_D(\lambda)}, \tag{5.17}$$

and it has zeros when $\lambda = \Lambda_{1,2}$. However, it is easy to see that $p^{(1)}(\lambda)$ has no pole at $\lambda = \Lambda_1$ as well as $q^{(1)}(\lambda)$ has no pole at $\lambda = \Lambda_2$. Thus the function $p^{(1)}(\lambda)$ (and, hence, the matrix $\hat{M}_-$) can be analytically continued into some vicinity of the real axis to the lower half-plane. Similarly the matrix $\hat{M}_+$ can be continued into a vicinity of the real axis to the upper half-plane. Hence, we again arrive at the contour Fig.1.

The following considerations are absolutely identical to the considerations of the previous section. Introducing the operator $\hat{U}(\lambda)$ by equations (4.25) (with $\hat{\Phi}$ replaced by $\hat{\Phi}^{(0)}$), we again obtain that all the jump matrices associated with the function $\hat{U}(\lambda)$ are exponentially small for the large value of t on the contour Fig.1 away from λ_0. Therefore, we have again that

$$\hat{U} \approx \hat{I}. \tag{5.18}$$

However, in distinction of the previous case, in order to find the original solution $\hat{\chi}$ we need to take into account the contribution of the operator $\hat{\Pi}$ (5.5).

The system of equations (5.6) gives

$$\begin{aligned}
\hat{\Pi}_{11} &= -\Lambda_1\hat{\imath} + \Lambda_{12}\hat{\Phi}^{(0)}_{11}(\Lambda_2)\hat{D}, \\
\hat{\Pi}_{12} &= \Lambda_{21}\hat{\Phi}^{(0)}_{11}(\Lambda_2)\hat{D}\ \hat{\Phi}^{(0)}_{12}(\Lambda_1)\left(\hat{\Phi}^{(0)}_{22}(\Lambda_1)\right)^{-1}, \\
\hat{\Pi}_{21} &= \Lambda_{12}\hat{\Phi}^{(0)}_{21}(\Lambda_2)\hat{D}, \\
\hat{\Pi}_{22} &= -\Lambda_1\hat{\imath} + \Lambda_{21}\hat{\Phi}^{(0)}_{21}(\Lambda_2)\hat{D}\ \hat{\Phi}^{(0)}_{12}(\Lambda_1)\left(\hat{\Phi}^{(0)}_{22}(\Lambda_1)\right)^{-1}.
\end{aligned} \tag{5.19}$$

where $\Lambda_{jk} = \Lambda_j - \Lambda_k$ and

$$\hat{D} = \left[\hat{\Phi}^{(0)}_{11}(\Lambda_2) - \hat{\Phi}^{(0)}_{12}(\Lambda_1)\left(\hat{\Phi}^{(0)}_{22}(\Lambda_1)\right)^{-1}\hat{\Phi}^{(0)}_{21}(\Lambda_2)\right]^{-1}. \tag{5.20}$$

The solution $\hat{\Phi}^{(0)}(\lambda)$ of the RHP (g) is asymptoticly equal to $\hat{M}_+^{-1}$ and $\hat{M}_-$ in the domains Ω_2 and Ω_3 respectively. Hence,

$$\hat{\Phi}^{(0)}(\Lambda_1) \approx \hat{M}_-(\Lambda_1) = \begin{pmatrix} \hat{\imath} & \hat{P}^{(1)}(\Lambda_1)e^{\tau(\Lambda_1)} \\ 0 & \hat{\imath} \end{pmatrix}, \tag{5.21}$$

$$\hat{\Phi}^{(0)}(\Lambda_2) \approx \hat{M}_+^{-1}(\Lambda_2) = \begin{pmatrix} \hat{\imath} & 0 \\ -\hat{Q}^{(1)}(\Lambda_2)e^{-\tau(\Lambda_2)} & \hat{\imath} \end{pmatrix}. \tag{5.22}$$

Thus, we obtain

$$\begin{aligned} &\hat{\Phi}^{(0)}_{11}(\Lambda_1),\ \hat{\Phi}^{(0)}_{11}(\Lambda_2),\ \hat{\Phi}^{(0)}_{22}(\Lambda_1),\ \hat{\Phi}^{(0)}_{22}(\Lambda_2) \approx \hat{\imath}, \\ &\hat{\Phi}^{(0)}_{21}(\Lambda_1),\ \hat{\Phi}^{(0)}_{12}(\Lambda_2) \approx 0, \\ &\hat{\Phi}^{(0)}_{21}(\Lambda_2) \approx -\hat{Q}^{(1)}(\Lambda_2)e^{-\tau(\Lambda_2)}, \\ &\hat{\Phi}^{(0)}_{12}(\Lambda_1) \approx \hat{P}^{(1)}(\Lambda_1)e^{\tau(\Lambda_1)}. \end{aligned} \tag{5.23}$$

Therefore, equations (5.19) can be replaced by the equations,

$$\begin{aligned} &\hat{\Pi}_{11} \approx -\Lambda_1\hat{\imath} + \Lambda_{12}\hat{D}, \qquad \hat{\Pi}_{12} \approx \Lambda_{21}\hat{D}\ \hat{\Phi}^{(0)}_{12}(\Lambda_1), \\ &\hat{\Pi}_{21} \approx \Lambda_{12}\hat{\Phi}^{(0)}_{21}(\Lambda_2)\hat{D}, \qquad \hat{\Pi}_{22} \approx -\Lambda_1\hat{\imath} + \Lambda_{21}\hat{\Phi}^{(0)}_{21}(\Lambda_2)\hat{D}\ \hat{\Phi}^{(0)}_{12}(\Lambda_1), \end{aligned} \tag{5.24}$$

and

$$\hat{D} \approx \left[\hat{\imath} - \hat{\Phi}^{(0)}_{12}(\Lambda_1)\hat{\Phi}^{(0)}_{21}(\Lambda_2)\right]^{-1}. \tag{5.25}$$

Define new vectors

$$\begin{aligned} |\overset{i}{1}\rangle &= \frac{\delta_\epsilon(u-\Lambda_i)Z(u,\Lambda_i)}{\sqrt{\mathcal{N}_\epsilon(\Lambda_i)Z(\Lambda_i,\Lambda_i)}}, \qquad & \langle\overset{i}{1}| &= \sqrt{\frac{Z(\Lambda_i,\Lambda_i)}{\mathcal{N}_\epsilon(\Lambda_i)}}Z(v,\Lambda_i), \\ |\overset{i}{2}\rangle &= \sqrt{\frac{Z(\Lambda_i,\Lambda_i)}{\mathcal{N}_\epsilon(\Lambda_i)}}Z(u,\Lambda_i), \qquad & \langle\overset{i}{2}| &= \frac{\delta_\epsilon(v-\Lambda_i)Z(v,\Lambda_i)}{\sqrt{\mathcal{N}_\epsilon(\Lambda_i)Z(\Lambda_i,\Lambda_i)}}, \end{aligned} \tag{5.26}$$

where $i = 1, 2$, and introduce two quantities

$$\alpha = \langle\overset{2}{1}|\, \varrho_+^{-1}(\Lambda_2)\varrho_-(\Lambda_1)|\overset{1}{1}\rangle\,, \tag{5.27}$$

$$\beta = \langle\overset{1}{2}|\, \varrho_-^T(\Lambda_1)\left(\varrho_+^T(\Lambda_2)\right)^{-1}|\overset{2}{2}\rangle\,. \tag{5.28}$$

Then it is easy to see that

$$\begin{aligned} &\hat{\Phi}^{(0)}_{21}(\Lambda_2) \approx -q^{(1)}(\Lambda_2)e^{-\tau(\Lambda_2)}\left(\hat{\varrho}_+^T(\Lambda_2)\right)^{-1}|\overset{2}{2}\rangle\langle\overset{2}{1}|\,\hat{\varrho}_+^{-1}(\Lambda_2), \\ &\hat{\Phi}^{(0)}_{12}(\Lambda_1) \approx p^{(1)}(\Lambda_1)e^{\tau(\Lambda_1)}\hat{\varrho}_-(\Lambda_1)|\overset{1}{1}\rangle\langle\overset{1}{2}|\,\hat{\varrho}_-^T(\Lambda_1), \end{aligned} \tag{5.29}$$

and hence

$$\hat{D} \approx \hat{\imath} - \frac{\beta p^{(1)}(\Lambda_1)q^{(1)}(\Lambda_2)e^{\tau(\Lambda_1)-\tau(\Lambda_2)}}{1+\alpha\beta p^{(1)}(\Lambda_1)q^{(1)}(\Lambda_2)e^{\tau(\Lambda_1)-\tau(\Lambda_2)}}\varrho_-(\Lambda_1)|\overset{1}{1}\rangle\langle\overset{2}{1}|\,\varrho_+^{-1}(\Lambda_2), \tag{5.30}$$

$$\hat{\Pi}_{11} \approx -\Lambda_2 \hat{\imath} + \frac{\Lambda_{21}\beta p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}}{1+\alpha\beta p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}} \varrho_-(\Lambda_1) |\overset{1}{1}\rangle\langle\overset{2}{1}|\, \varrho_+^{-1}(\Lambda_2), \tag{5.31}$$

$$\hat{\Pi}_{12} \approx \frac{\Lambda_{21} p^{(1)}(\Lambda_1) e^{\tau(\Lambda_1)}}{1+\alpha\beta p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}} \varrho_-(\Lambda_1) |\overset{1}{1}\rangle\langle\overset{1}{2}|\, \varrho_-^{T}(\Lambda_1). \tag{5.32}$$

Now, let us imtroduce the coefficients of the asymptotic expansion

$$\hat{\Phi}^{(0)}(\lambda) = \hat{I} + \frac{1}{\lambda}\hat{B}^{(0)} + \frac{1}{\lambda^2}\hat{C}^{(0)} \dots, \tag{5.33}$$

Since $\hat{\Phi}^{(0)} \approx \hat{I}$ for complex λ, we have that $\hat{B}^{(0)}, \hat{C}^{(0)} \approx 0$. Using expansion (5.33) and expansion (4.9) for the function $\hat{\varrho}$, we obtain the following asymptotic representations for the operators $\hat{b}_{kj}$ and $\hat{c}_{kj}$,

$$\hat{b}_{11} = \hat{\Pi}_{11} + \hat{\varrho}_0 + \Lambda_2\hat{\imath} + o(1), \tag{5.34}$$

$$\hat{c}_{22} - \hat{c}_{11} = (\Lambda_1\hat{\imath} + \hat{\Pi}_{22})(\Lambda_1\hat{\imath} - \hat{\varrho}_0^T) - (\Lambda_2\hat{\imath} + \hat{\Pi}_{11})(\Lambda_2\hat{\imath} + \hat{\varrho}_0) - \hat{\varrho}_1 - \hat{\varrho}_1^T + \left(\hat{\varrho}_0^T\right)^2 + o(1), \tag{5.35}$$

and

$$\hat{b}_{12} = \hat{\Pi}_{12} + o(1). \tag{5.36}$$

The logarithmic derivatives of the Fredholm determinant with respect to x and t (i.e. λ_0 and t) are equal to the traces of the operators (5.34) and (5.35) respectively. Consider more detailed the first one. From (5.31) and (5.34) we have

$$\hat{b}_{11} = \hat{\varrho}_0 + \frac{\beta\Lambda_{21} p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}}{1+\alpha\beta p^{(1)}(\Lambda_1) Q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}} \hat{\varrho}_-(\Lambda_1) |\overset{1}{1}\rangle\langle\overset{2}{1}|\, \hat{\varrho}_+^{-1}(\Lambda_2) + o(1) \tag{5.37}$$

hence

$$\begin{aligned} \operatorname{tr}\hat{b}_{11} &= \operatorname{tr}\hat{\varrho}_0 + \frac{\alpha\beta\Lambda_{21} p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}}{1+\alpha\beta p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}} + o(1) \\ &= \operatorname{tr}\hat{\varrho}_0 - i\frac{\partial}{\partial x} \ln\left(1+\alpha\beta p^{(1)}(\Lambda_1) q^{(1)}(\Lambda_2) e^{\tau(\Lambda_1)-\tau(\Lambda_2)}\right) + o(1). \end{aligned} \tag{5.38}$$

The trace of the operator $\hat{\varrho}_0$ as before is equal to the first coefficient of the asymptotic expansion of its determinant Δ_0 (see (4.11)), which due to (5.4) is equal to

$$\Delta_0 = \frac{1}{2\pi i}\int_\Gamma d\mu\, \operatorname{sign}(\lambda_0 - \Re\mu) \ln\left(1 - \vartheta(\mu)\left(1 + e^{\phi(\mu)\operatorname{sign}(\Re\mu-\lambda_0)}\right)\right). \tag{5.39}$$

In comparison with (4.38) the difference is that one should integrate now with respect to deformed contour Γ instead of the real axis.

Similarly one can find the trace of the operator $\hat{c}_{22} - \hat{c}_{11}$. We do not present here the details of these calculations; they are quite strightforward, although rather long. The resulting formula for

the leading term of the asymptotics of the Fredholm determinant is

$$\det(\tilde{I}+\tilde{V}) = \Big(1+\alpha\beta p^{(1)}(\Lambda_1)q^{(1)}(\Lambda_2)e^{\tau(\Lambda_1)-\tau(\Lambda_2)}\Big)$$

$$\times \exp\left\{\frac{1}{2\pi}\int_\Gamma d\mu|x-2\mu t|\ln\Big(1-\vartheta(\mu)\Big(1+e^{\phi(\mu)\,\mathrm{sign}(\Re\mu-\lambda_0)}\Big)\Big)\right\}. \tag{5.40}$$

The symbol $|x-2\mu t|$ is understood according to the equation,

$$|x-2\mu t| \equiv (x-2\mu t)\,\mathrm{sign}(\Re\mu-\lambda_0).$$

We would like to draw attention of the reader to the difference between the cases of positive and negative chemical potential. In the last case the determinants of the operators $\hat{G}_{11}$ and $\hat{G}_{22}$

$$\det\hat{G}_{11(22)}(\lambda) = 1-\vartheta(\lambda)\Big(1+e^{-(+)\phi(\lambda)}\Big) \tag{5.41}$$

have no zeros at the real axis, and the zero-index condition,

$$\arg\det\Big(\theta(\lambda_0-\lambda)\hat{G}_{11}(\lambda)+\theta(\lambda-\lambda_0)(\hat{G}_{22}^T)^{-1}(\lambda)\Big)\to 0,\quad |\lambda|\to\infty, \tag{5.42}$$

takes place on the real line. Thus the integral in (4.41) is well defined. In the case of positive chemical potential $\det\hat{G}_{11}$ have real roots, and therefore one should choose the integration contour more accurately in order to satisfy (5.42). We have seen, that this means that the real axis have to be replaced by the contour Γ.

The second important difference, is that the leading term of the solution of the RHP (a) is diagonal in the case of negative chemical potential (4.31). Thus the coefficient $\hat{b}_{12}$ is asymptoticly equal to zero (in the next section we will see that actually $\hat{b}_{12}=\mathcal{O}(t^{-1/2})$). This means that in the case of negative chemical potential the accuracy we reached so far is not enough for evaluation of the full combination $\int dudv\hat{b}_{12}\cdot\det\Big(\tilde{I}+\widetilde{V}\Big)$, which describes the correlation function. On the contrary in the case of positive chemical potential we obtain non-zero value for $\hat{b}_{12}$ already in the leading term of the asymptotics,

$$\hat{b}_{12} = \frac{\Lambda_{21}p^{(1)}(\Lambda_1)e^{\tau(\Lambda_1)}}{1+\alpha\beta p^{(1)}(\Lambda_1)q^{(1)}(\Lambda_2)e^{\tau(\Lambda_1)-\tau(\Lambda_2)}}\hat{\varrho}_-(\Lambda_1)|\overset{1}{1}\rangle\langle\overset{1}{2}|\,\hat{\varrho}_-^T(\Lambda_1) \tag{5.43}$$

$$+o(1).$$

Observe, that the pre-exponent factor in (5.40) cancels the denominator in (5.43), so we obtain

$$\mathcal{B} = \Lambda_{21}p^{(1)}(\Lambda_1)e^{\tau(\Lambda_1)}\int_{-\infty}^{\infty}dudv\,\hat{\varrho}_-(\Lambda_1)|\overset{1}{1}\rangle\langle\overset{1}{2}|\,\hat{\varrho}_-^T(\Lambda_1)$$

$$\times \exp\left\{\frac{1}{2\pi}\int_\Gamma d\mu|x-2\mu t|\ln\Big(1-\vartheta(\mu)\Big(1+e^{\phi(\mu)\,\mathrm{sign}(\Re\mu-\lambda_0)}\Big)\Big)\right\}. \tag{5.44}$$

As we have mentioned already, the different asymptotics for negative and positive chemical potential has deep physical origin. This difference becomes essentially important in the low-temperature limit.

6 The localized RHP

The estimates of the previous sections are not enough for comprehensive analysis of the correlation function (1.1). In particular, we saw that for the case of negative chemical potential the leading term of the asymptotics of the RHP (a) solution is diagonal, therefore we can not find the factor $\hat{b}_{12}$, which is necessary for study of the correlation function. In order to improve the estimates obtained in the previous sections one needs to perform the second step of the nonlinear steepest descent method, i.e. to consider the RHP in the vicinity of the saddle point—so called localized Riemann–Hilbert problem. Setting $\hat{U} \approx \hat{I}$ in (4.31) we neglect the contribution of the vicinity $|\lambda - \lambda_0| < t^{-1/2}$. Now we consider this problem in more detail. The difference between positive and negative chemical potential now is not very essential, therefore for simplicity we concentrate our attention at the negative chemical potential only.

Since the vicinity of the saddle point $|\lambda - \lambda_0| < t^{-1/2}$ is small for large t one could replace jump matrices $\hat{N}_\pm(\lambda)$ and $\hat{M}_\pm(\lambda)$ by their values in the point λ_0: $\hat{N}_\pm(\lambda_0)$ and $\hat{M}_\pm(\lambda_0)$. However, $\hat{N}_\pm(\lambda)$ and $\hat{M}_\pm(\lambda)$ are well defined only in the vicinity of λ_0, but not exactly in the saddle point. Indeed, all of them depend on $\hat{\varrho}$ (see (4.17), (4.18), (4.20), (4.21), (4.23) and (4.24)). In turn, $\hat{\varrho}$ satisfies RHP (b) with jump operator having a discontinuity in the saddle point. Recall the jump condition for the boundary values of $\hat{\varrho}$:

$$\hat{\varrho}_- = \hat{\varrho}_+ \hat{\mathcal{D}} \qquad \lambda \in R. \tag{6.1}$$

where

$$\hat{\mathcal{D}} = \theta(\lambda_0 - \lambda)\hat{G}_{11} + \theta(\lambda - \lambda_0)(\hat{G}_{22}^T)^{-1}. \tag{6.2}$$

It is easy to see that

$$\hat{\mathcal{D}}_l \equiv \lim_{\lambda \to \lambda_0 - 0} \hat{\mathcal{D}}(\lambda) \neq \hat{\mathcal{D}}_r \equiv \lim_{\lambda \to \lambda_0 + 0} \hat{\mathcal{D}}(\lambda). \tag{6.3}$$

Thus, the solution $\hat{\varrho}$ of the RHP (b) has a branch point at $\lambda = \lambda_0$.

The explicit expression for jump operator $\hat{\mathcal{D}}$ is given by (3.11), (6.2). It has the following structure

$$\hat{\mathcal{D}} = \theta(\lambda_0 - \lambda)(\hat{\imath} + f_1(\lambda)\hat{\omega}) + \theta(\lambda - \lambda_0)(\hat{\imath} + f_2(\lambda)\hat{\omega}). \tag{6.4}$$

Here $f_1(\lambda)$ and $f_2(\lambda)$ are equal to

$$f_1(\lambda) = -\vartheta(\lambda) Z(\lambda, \lambda) e^{\phi_D(\lambda)},$$
$$f_2(\lambda) = \frac{\vartheta(\lambda) Z(\lambda, \lambda) e^{\phi_A(\lambda)}}{1 - Z(\lambda, \lambda)\vartheta(\lambda) e^{\phi_A(\lambda)}}, \tag{6.5}$$

and

$$\hat{\omega}(\lambda) \equiv \hat{\omega}(\lambda | u, v) = |1\rangle\langle 1|. \tag{6.6}$$

The operator $\hat{\omega}$ will play an important role. Obviously $\hat{\omega}$ is one-dimensional projector with the trace equal to 1, hence,

$$\hat{\omega}^n = \hat{\omega}. \tag{6.7}$$

Defining an operator-valued function $(\lambda - \lambda_0)^{\xi\hat{\omega}}$, where ξ is a number, as a formal Taylor series with respect to $\hat{\omega}$ we obtain

$$(\lambda - \lambda_0)^{\xi\hat{\omega}} = \hat{\imath} + \hat{\omega}\left((\lambda - \lambda_0)^{\xi} - 1\right) \tag{6.8}$$

We shall consider formula (6.8) as non-formal definition of the operator-valued function $(\lambda - \lambda_0)^{\xi\hat{\omega}}$. We will also assume that the branch of the function $(\lambda - \lambda_0)^{\xi}$ is fixed by the condition,

$$-\pi < \arg(\lambda - \lambda_0) < \pi. \tag{6.9}$$

It is easy to check the following properties

$$\left((\lambda - \lambda_0)^{\xi\hat{\omega}}\right)^{-1} = (\lambda - \lambda_0)^{-\xi\hat{\omega}} = \hat{\imath} + \hat{\omega}\left((\lambda - \lambda_0)^{-\xi} - 1\right), \tag{6.10}$$

$$(\lambda - \lambda_0)_+^{\xi\hat{\omega}}(\lambda - \lambda_0)_-^{-\xi\hat{\omega}} = \hat{\imath} + \theta(\lambda_0 - \lambda)\hat{\omega}\left(e^{2\pi i\xi} - 1\right) = \theta(\lambda_0 - \lambda)e^{2\pi i\xi\hat{\omega}} + \theta(\lambda - \lambda_0)\hat{\imath}, \tag{6.11}$$

Put

$$\hat{\varrho}(\lambda) = \hat{\varrho}^{(c)}(\lambda)(\lambda - \lambda_0)^{is\hat{\omega}_0}, \tag{6.12}$$

where $\hat{\omega}_0 = \hat{\omega}(\lambda_0)$ and s is a complex parameter. Then jump condition for $\hat{\varrho}^{(c)}$ takes the form

$$\hat{\varrho}_-^{(c)} = \hat{\varrho}_+^{(c)}\hat{\mathcal{D}}_0, \tag{6.13}$$

where

$$\hat{\mathcal{D}}_0(\lambda) = (\lambda - \lambda_0)_+^{is\hat{\omega}_0}\hat{\mathcal{D}}(\lambda)(\lambda - \lambda_0)_-^{-is\hat{\omega}_0}. \tag{6.14}$$

We want equation (6.12) to represent the singularity of $\hat{\varrho}(\lambda)$ near the saddle point λ_0, i.e. we want the $\pm$-values of the factor $\hat{\varrho}^{(c)}(\lambda)$ to be continuous at λ_0. To ensure this property we demand that $\hat{\mathcal{D}}_0(\lambda)$ has no jump at λ_0 (cf. the analysis of the usual RHPs near the points of the discontinuities of the jump coefficients [40]). Observe that the limit values of $\hat{\mathcal{D}}(\lambda)$ in the saddle point $\hat{\mathcal{D}}_l$ and $\hat{\mathcal{D}}_r$ commute with $(\lambda - \lambda_0)^{is\hat{\omega}_0}$. Thus, using (6.11) we have

$$\begin{aligned} \lim_{\lambda\to\lambda_0-0}\hat{\mathcal{D}}_0(\lambda) &= e^{-2\pi s\hat{\omega}_0}(\hat{\imath} + f_1(\lambda_0)\hat{\omega}_0), \\ \lim_{\lambda\to\lambda_0+0}\hat{\mathcal{D}}_0(\lambda) &= \hat{\imath} + f_2(\lambda_0)\hat{\omega}_0. \end{aligned} \tag{6.15}$$

Making right-hand sides of (6.15) equal to each other we find

$$s = \frac{1}{2\pi}\ln\frac{1 + f_1(\lambda_0)}{1 + f_2(\lambda_0)}, \tag{6.16}$$

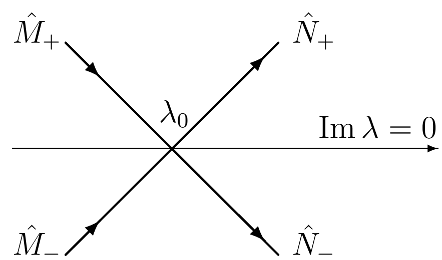

Figure 3: Jump contour and jump matrices in the vicinity of the saddle point

or, using (6.5)

$$s = \frac{1}{2\pi} \ln \left[\left(1 - \vartheta_0 Z_0 e^{\phi_D(\lambda_0)}\right) \left(1 - \vartheta_0 Z_0 e^{\phi_A(\lambda_0)}\right) \right], \tag{6.17}$$

where

$$\vartheta_0 = \vartheta(\lambda_0), \qquad Z_0 = Z(\lambda_0, \lambda_0). \tag{6.18}$$

Thus, we have found the behavior of $\hat{\varrho}$ in the saddle point. It is given by (6.12), where $\hat{\varrho}^{(c)}$ has the limits as λ approaches λ_0 from the upper and from the lower half planes, operator $\hat{\omega}$ is defined in (6.6), s is given by (6.17).

Let us turn back now to the RHP (d) (4.26) for the operator $\hat{U}$. We consider this problem in the vicinity of λ_0. The corresponding jump contour is shown on the Fig.3. It consists of four branches. One can continue the contour on the Fig.3 from the vicinity $|\lambda - \lambda_0| < t^{-1/2}$ to infinity and consider new RHP with jump conditions on two infinite axes. Such a replacement, of course, gives a non-zero contribution in the solution, but this contribution is exponentially small due to the presence of factors $e^{\pm\tau}$ in the jump matrices (cf. the classical analysis of the oscillatory integrals).

Let us introduce a notation $\hat{\mathcal{W}}$, which will denote matrices M_+, M_-, N_+ or N_- for the corresponding branches of the contour. Due to the fact that the functions $\hat{\varrho}^{(c)}_\pm(\lambda)$ are continuous at λ_0, we replace $\hat{\mathcal{W}}$ by $\hat{\mathcal{W}}_0 = \{\hat{M}^{(0)}_\pm, \quad \hat{N}^{(0)}_\pm\}$, where

$$\hat{M}^{(0)}_+ = \begin{pmatrix} \hat{\imath} & 0 \\ (\hat{\varrho}_0^{(c)T})^{-1}(\lambda-\lambda_0)^{-is\hat{\omega}_0^T} \cdot \hat{Q}^{(0)} \cdot (\lambda-\lambda_0)^{-is\hat{\omega}_0}(\hat{\varrho}_0^{(c)})^{-1} e^{-\tau} & \hat{\imath} \end{pmatrix}, \tag{6.19}$$

$$\hat{N}^{(0)}_- = \begin{pmatrix} \hat{\imath} & 0 \\ (\hat{\varrho}_0^{(c)T})^{-1}(\lambda-\lambda_0)^{-is\hat{\omega}_0^T} \cdot \hat{\tilde{Q}}^{(0)} \cdot (\lambda-\lambda_0)^{-is\hat{\omega}_0}(\hat{\varrho}_0^{(c)})^{-1} e^{-\tau} & \hat{\imath} \end{pmatrix}. \tag{6.20}$$

$$\hat{M}_{-}^{(0)} = \begin{pmatrix} \hat{\imath} & \hat{\varrho}_0^{(c)}(\lambda-\lambda_0)^{is\hat{\omega}_0} \cdot \hat{P}^{(0)} \cdot (\lambda-\lambda_0)^{is\hat{\omega}_0^T} \hat{\varrho}_0^{(c)T} e^{\tau} \\ 0 & \hat{\imath} \end{pmatrix}, \tag{6.21}$$

$$\hat{N}_{+}^{(0)} = \begin{pmatrix} \hat{\imath} & \hat{\varrho}_0^{(c)}(\lambda-\lambda_0)^{is\hat{\omega}_0} \cdot \hat{\tilde{P}}^{(0)} \cdot (\lambda-\lambda_0)^{is\hat{\omega}_0^T} \hat{\varrho}_0^{(c)T} e^{\tau} \\ 0 & \hat{\imath} \end{pmatrix}. \tag{6.22}$$

Here $\hat{\varrho}_0^{(c)} = \hat{\varrho}^{(c)}(\lambda_0)$, and

$$\hat{Q}^{(0)} e^{-\tau(\lambda_0)} = \hat{G}_{21}(\lambda_0)(\hat{G}_{11}(\lambda_0))^{-1}, \quad \hat{P}^{(0)} e^{\tau(\lambda_0)} = (\hat{G}_{11}(\lambda_0))^{-1}\hat{G}_{12}(\lambda_0) \tag{6.23}$$

$$\hat{\tilde{Q}}^{(0)} e^{-\tau(\lambda_0)} = (\hat{G}_{22}(\lambda_0))^{-1}\hat{G}_{21}(\lambda_0), \quad \hat{\tilde{P}}^{(0)} e^{\tau(\lambda_0)} = \hat{G}_{12}(\lambda_0)(\hat{G}_{22}(\lambda_0))^{-1}. \tag{6.24}$$

Strictly speaking, one should add the subscript '$_+$' to $\hat{\varrho}_0^{(c)}$ which appears in the matrices $\hat{M}_{+}^{(0)}$, $\hat{N}_{+}^{(0)}$ and the subscript '$_-$' to $\hat{\varrho}_0^{(c)}$ which appears in the matrices $\hat{M}_{-}^{(0)}$, $\hat{N}_{-}^{(0)}$. Unless the distinction is important, we usually will omit the subscripts to avoid the overcomplication of the notations.

Based on the experience with the classical NLS (see [39] and [37]), we expect that the replacement $\mathcal{W} \to \mathcal{W}_0$ will enable us to capture the terms of order $t^{-1/2}$ in the asymptotic solution of the original operator-valued RHP. These terms, as it has already been explained in the introduction, are of the most importance for the comprehensive asymptotic analysis of the correlation function (1.1).

Replacing $\mathcal{W} \to \mathcal{W}_0$, we approximate the exact solution $\hat{U}(\lambda)$ by the operator-valued function $\hat{U}^{(0)}(\lambda)$ which solves the following localized RHP (cf. again [39] and [37]),

$$\begin{aligned} &1^h.\ \hat{U}^{(0)}(\lambda) \to \hat{I} \quad \lambda \to \infty, \\ &2^h.\ \hat{U}^{(0)}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin C, \\ &3^h.\ \hat{U}_{-}^{(0)} = \hat{U}_{+}^{(0)}\hat{\mathcal{W}}_0, \qquad \lambda \in C, \end{aligned} \tag{6.25}$$

where C is the contour depicted in Fig.3. In this and the next two sections we will construct an explicit solution of this problem.

Making the substitution

$$\hat{U}^{(0)} = \hat{\Psi}\hat{S}, \tag{6.26}$$

where

$$\hat{S} = \begin{pmatrix} (\lambda-\lambda_0)^{-is\hat{\omega}_0}(\hat{\varrho}_0^{(c)})^{-1} e^{-\tau(\lambda)/2} & 0 \\ 0 & (\lambda-\lambda_0)^{is\hat{\omega}_0^T} \hat{\varrho}_0^{(c)T} e^{\tau(\lambda)/2} \end{pmatrix} \tag{6.27}$$

we obtain new RHP for the operator $\hat{\Psi}$:

$$1^i.\ \hat{\Psi}(\lambda) \to \hat{S}^{-1} \quad \lambda \to \infty,$$

$$2^i.\ \hat{\Psi}(\lambda) \quad \text{is analytical function of } \lambda \text{ if } \lambda \notin C \cup R, \tag{6.28}$$

$$3^{i_1}.\ \hat{\Psi}_- = \hat{\Psi}_+ \hat{\mathcal{W}}_\Psi, \qquad \lambda \in C, \tag{6.29}$$

where

$$\hat{\mathcal{W}}_\Psi = \hat{S}\hat{\mathcal{W}}_0\hat{S}^{-1} = \{\hat{M}^{(0)}_{\pm,\Psi}, \quad \hat{N}^{(0)}_{\pm,\Psi}\}, \tag{6.30}$$

and

$$\hat{M}^{(0)}_{+,\Psi} = \begin{pmatrix} \hat{\imath} & 0 \\ \hat{Q}_0 & \hat{\imath} \end{pmatrix}, \qquad \hat{N}^{(0)}_{-,\Psi} = \begin{pmatrix} \hat{\imath} & 0 \\ \hat{\tilde{Q}}^{(0)} & \hat{\imath} \end{pmatrix}, \tag{6.31}$$

$$\hat{M}^{(0)}_{-,\Psi} = \begin{pmatrix} \hat{\imath} & \hat{P}_0 \\ 0 & \hat{\imath} \end{pmatrix}, \qquad \hat{N}^{(0)}_{+,\Psi} = \begin{pmatrix} \hat{\imath} & \hat{\tilde{P}}^{(0)} \\ 0 & \hat{\imath} \end{pmatrix}. \tag{6.32}$$

Since the matrix $\hat{S}$ depends on $(\lambda-\lambda_0)^{is\hat{\omega}_0}$ and $\hat{\varrho}_0^{(c)}$, we have an additional jumps on the real axis

$$3^{i_2}.\ \hat{\Psi}_- = \hat{\Psi}_+ \hat{K}, \qquad \lambda \in R \tag{6.33}$$

where

$$\hat{K} \equiv \hat{K}_+ = \begin{pmatrix} \hat{\imath} + f_1(\lambda_0)\hat{\omega}_0 & 0 \\ 0 & (\hat{\imath} + f_1(\lambda_0)\hat{\omega}_0^T)^{-1} \end{pmatrix}, \qquad \lambda < \lambda_0, \tag{6.34}$$

$$\hat{K} \equiv \hat{K}_- = \begin{pmatrix} \hat{\imath} + f_2(\lambda_0)\hat{\omega}_0 & 0 \\ 0 & (\hat{\imath} + f_2(\lambda_0)\hat{\omega}_0^T)^{-1} \end{pmatrix}, \qquad \lambda > \lambda_0, \tag{6.35}$$

For the functions $f_{1,2}(\lambda)$ see (6.5).

It is convenient to make the following change of the variable λ,

$$\lambda \to z: \quad \lambda - \lambda_0 = z e^{i\frac{\pi}{4}},$$

so that,

$$\hat{S} = \begin{pmatrix} z^{-is\hat{\omega}_0} e^{\pi s\hat{\omega}_0/4} (\hat{\varrho}_0^{(c)})^{-1} e^{\frac{tz^2}{2} + \frac{it\lambda_0^2}{2}} & 0 \\ 0 & z^{is\hat{\omega}_0^T} e^{-\pi s\hat{\omega}_0^T/4} \hat{\varrho}_0^{(c)T} e^{-\frac{tz^2}{2} - \frac{it\lambda_0^2}{2}} \end{pmatrix} \tag{6.36}$$

Note that matrices $\hat{\mathcal{W}}_\Psi$ and $\hat{K}$ do not depend on λ, hence they do not change.

Let $\hat{\kappa}$ be the first coefficient of the asymptotic expansion of $\hat{U}^{(0)}$:

$$\hat{U}^{(0)} = \hat{I} + \frac{\hat{\kappa}}{\lambda} + \dots. \tag{6.37}$$

Then the asymptotics of $\hat{\Psi}$ is given by the expression

$$\hat{\Psi} = (\hat{I} + \frac{\hat{\kappa}}{z} e^{-i\frac{\pi}{4}} + \ldots)\hat{S}^{-1}. \tag{6.38}$$

Due to the property that all the jump matrices do not depend on z we conclude that $(\partial_z \hat{\Psi})\hat{\Psi}^{-1}$ is holomorphic in the finite complex plane. The asymptotics of this 'logarithmic derivative' $(\partial_z \hat{\Psi})\hat{\Psi}^{-1}$ can be found from equality

$$(\partial_z \hat{\Psi})\hat{\Psi}^{-1} = (\partial_z \hat{U}^{(0)})\hat{U}^{(0)-1} - \hat{U}^{(0)}\hat{S}^{-1}\partial_z \hat{S}\hat{U}^{(0)-1}. \tag{6.39}$$

It is easy to see that

$$\hat{S}^{-1}\partial_z \hat{S} = \begin{pmatrix} tz\hat{\imath} - \dfrac{is}{z}\hat{\omega}_0 & 0 \\ 0 & -tz\hat{\imath} + \dfrac{is}{z}\hat{\omega}_0^T \end{pmatrix}, \tag{6.40}$$

therefore

$$(\partial_z \hat{\Psi})\hat{\Psi}^{-1} \to -tz\hat{\sigma}_3 + \mathcal{O}(1), \qquad z \to \infty. \tag{6.41}$$

Due to the Liouville theorem $(\partial_z \hat{\Psi})\hat{\Psi}^{-1}$ is a first order polynomial. Using (6.38) we find

$$(\partial_z \hat{\Psi})\hat{\Psi}^{-1} = -tz\hat{\sigma}_3 - t[\hat{\kappa}, \hat{\sigma}_3]e^{-i\frac{\pi}{4}}, \tag{6.42}$$

and, hence, $\hat{\Psi}$ satisfies ordinary differential equation

$$\partial_z \hat{\Psi} = -t(z\hat{\sigma}_3 + [\hat{\kappa}, \hat{\sigma}_3]e^{-i\frac{\pi}{4}})\hat{\Psi}. \tag{6.43}$$

The arguments which led us to the differential equation (6.43) are identical to the arguments used in the theory of classical integrable systems. Similar to the pure matrix case [39], [37], [41], the first order 'matrix' differential equation for $\hat{\Psi}$ implies the second order differential equation for components $\hat{\Psi}_{jk}$. After one more replacement, $\xi = z\sqrt{2t}$, we obtain, for example, for Ψ_{11}:

$$\frac{d^2}{d\xi^2}\hat{\Psi}_{11} + \left(\frac{1}{2} - \frac{\xi^2}{4}\right)\hat{\Psi}_{11} + \hat{\nu}\hat{\Psi}_{11} = 0. \tag{6.44}$$

Here $\hat{\nu} = -2it\hat{\kappa}_{12}\hat{\kappa}_{21}$. This equation looks like the parabolic cylinder equation, except that $\hat{\nu}$ *is an operator*. In the next section we shall give rigorous definition of solution of equation (6.44). At the moment one can consider the parabolic cylinder function (PCF) $D_{\hat{\nu}}(\xi)$ with operator-valued index $\hat{\nu}$ as a formal Tailor series with respect to $\hat{\nu}$. This series satisfies recurrence, which is valid for usual PCF:

$$\begin{aligned} &\frac{d}{d\xi}D_\nu(\xi) + \frac{\xi}{2}D_\nu(\xi) - \hat{\nu}D_{\nu-1}(\xi) = 0, \\ &\frac{d}{d\xi}D_\nu(\xi) - \frac{\xi}{2}D_\nu(\xi) + D_{\nu+1}(\xi) = 0, \end{aligned} \tag{6.45}$$

Using these properties, one can check directly that the following operator-valued matrix

$$\hat{\Psi} = \begin{pmatrix} D_{\hat{\nu}}(\xi) & \sqrt{2t}e^{i\frac{\pi}{4}}\hat{\kappa}_{12}D_{\hat{\tilde{\nu}}-\hat{\imath}}(i\xi) \\ \sqrt{2t}e^{-i\frac{\pi}{4}}\hat{\kappa}_{21}D_{\hat{\nu}-\hat{\imath}}(\xi) & D_{\hat{\tilde{\nu}}}(i\xi) \end{pmatrix} \hat{L}. \tag{6.46}$$

satisfies equation (6.43). Here

$$\hat{\nu} = -2it\hat{\kappa}_{12}\hat{\kappa}_{21}, \qquad \hat{\tilde{\nu}} = 2it\hat{\kappa}_{21}\hat{\kappa}_{12}, \tag{6.47}$$

and $\hat{L}$ is piecewise constant operator-valued matrix.

7 The parabolic cylinder functions with an operator index

To complete the solution of the model RHP (h), or equivalently the RHP (i) (see (6.28) - (6.32)), the quantaties $\hat{\kappa}_{jl}$ and $\hat{L}$ involved in (6.46) are needed to be determined explicitly in terms of the jump matrix $\hat{\mathcal{W}}_{\Psi}$. Also, it follows from the comparision of expansion (6.37) for $\hat{U}^{(0)}$ and expansion (2.16) for original operator $\hat{\chi}$ that

$$\hat{\kappa}_{12} = \hat{b}_{12} \quad \text{and} \quad \hat{\kappa}_{21} = \hat{b}_{21}. \tag{7.1}$$

Thus, in order to obtain the asymptotics of the correlation function we need to find just these operators.

To achieve the objectives formulated above, we have to make the PCF $D_{\hat{\nu}}(\xi)$ with the operator-valued index $\hat{\nu}$ a non-formal object. In fact, we need to control its asymptotic behavior as $\xi \to \infty$ everywhere on the complex plane ξ. In other words, similar to the classical PCF, we want $D_{\hat{\nu}}(\xi)$ to be a genuine special function. To this end we need to know more about some special features of the operators $\hat{\kappa}_{jl}$ and $\hat{\nu}$ which follow from the setting of the RHP(i). In this and the next sections we show that the structure of the jump matrices of the model RHP(i) suggests a very special anzats for these operators which allows eventually to reduce the operator-valued problem (i) to a matrix one. This in turn will give us the possibility to obtain the explicit expressions for $\hat{\kappa}_{jl}$.

In the previous section we have already replaced variable λ by variable ξ:

$$\lambda - \lambda_0 = \frac{\xi}{\sqrt{2t}}e^{i\frac{\pi}{4}}. \tag{7.2}$$

Further we shall work with variable ξ only. Therefore we present here jump contours in the complex plane of ξ. The complex ξ-plane consists of six sectors $I, \ldots, VI$. The boundaries of the sectors $C_{i-j}, \quad i,j = I, \ldots, VI$ are shown on the Fig.4. They are jump contours of the matrix $\hat{\Psi}$. The

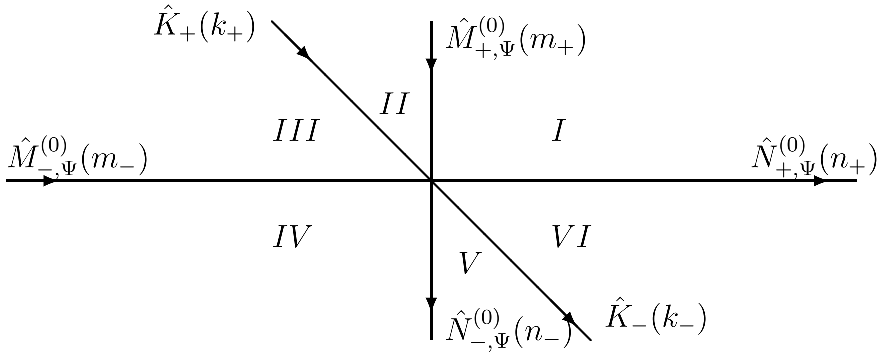

Figure 4: Jump contour and jump matrices in the "ξ"-plane

corresponding operator-valued jump matrices also placed on the Fig.4. In the parenthesis we have placed scalar analogs of jump matrices, which will be discussed later.

The piecewise constant matrix $\hat{L}$ and operators $\hat{\kappa}_{12}$ and $\hat{\kappa}_{21}$ are uniquely defined by the condition that matrix operator $\hat{\Psi}$ (6.46) satisfies the jump equations indicated in (6.28)-(6.32) and possesses asymptotics (6.38). These restrictions should provide us the explicit expressions for $\hat{\kappa}_{jk}$ and $\hat{L}$.

In what follows we shall consider vectors $|1\rangle$, $|2\rangle$, $\langle 1|$ and $\langle 2|$ only in the point $\lambda = \lambda_0$. Therefore, we introduce new objects

$$\begin{aligned} |\mathring{1}\rangle &= \frac{\delta_\epsilon(u-\lambda_0)Z(u,\lambda_0)}{\sqrt{\mathcal{N}_\epsilon(\lambda_0)Z_0}}, & \langle\mathring{1}| &= \sqrt{\frac{Z_0}{\mathcal{N}_\epsilon(\lambda_0)}}Z(v,\lambda_0), \\ |\mathring{2}\rangle &= \sqrt{\frac{Z_0}{\mathcal{N}_\epsilon(\lambda_0)}}Z(v,\lambda_0), & \langle\mathring{2}| &= \frac{\delta_\epsilon(v-\lambda_0)Z(v,\lambda_0)}{\sqrt{\mathcal{N}_\epsilon(\lambda_0)Z_0}}. \end{aligned} \tag{7.3}$$

Recall once more that $Z_0 = Z(\lambda_0,\lambda_0)$. Denoting as before $\hat{\omega}(\lambda_0) = \hat{\omega}_0$, we have

$$\hat{\omega}_0 = |\mathring{1}\rangle\langle\mathring{1}|\,, \qquad \hat{\omega}_0^T = |\mathring{2}\rangle\langle\mathring{2}|\,. \tag{7.4}$$

Using the Taylor series definition, we obtain for an arbitrary holomorphic at the origin function $f(x)$:

$$\begin{aligned} f\left(x|\mathring{1}\rangle\langle\mathring{1}|\right) &= f(0)\left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}|\right) + |\mathring{1}\rangle\langle\mathring{1}|\, f(x), \\ f\left(x|\mathring{2}\rangle\langle\mathring{2}|\right) &= f(0)\left(\hat{\imath} - |\mathring{2}\rangle\langle\mathring{2}|\right) + |\mathring{2}\rangle\langle\mathring{2}|\, f(x), \end{aligned} \tag{7.5}$$

$$\begin{aligned} |\mathring{2}\rangle\langle\mathring{1}|\, f\left(x|\mathring{1}\rangle\langle\mathring{1}|\right) &= f(x)|\mathring{2}\rangle\langle\mathring{1}|\,, \\ |\mathring{1}\rangle\langle\mathring{2}|\, f\left(x|\mathring{2}\rangle\langle\mathring{2}|\right) &= f(x)|\mathring{1}\rangle\langle\mathring{2}|\,. \end{aligned} \tag{7.6}$$

Note also the following generalization of (7.5),

$$f\left(a\hat{\imath}+x|\mathring{1}\rangle\langle\mathring{1}|\right)=f(a)\left(\hat{\imath}-|\mathring{1}\rangle\langle\mathring{1}|\right)+|\mathring{1}\rangle\langle\mathring{1}|\,f(x+a),$$
$$f\left(a\hat{\imath}+x|\mathring{2}\rangle\langle\mathring{2}|\right)=f(a)\left(\hat{\imath}-|\mathring{2}\rangle\langle\mathring{2}|\right)+|\mathring{2}\rangle\langle\mathring{2}|\,f(x+a), \tag{7.7}$$

which takes place for a function $f(x)$ holomorphic at the point $x=a$.

Now, we are ready to describe our anzats. The structure of jump matrices $\hat{N}^{(0)}_{\pm,\Psi}$ and $\hat{M}^{(0)}_{\pm,\Psi}$ allows us to look for the operators $\hat{\kappa}_{12}$ and $\hat{\kappa}_{21}$ in the form of one-dimensional projectors. Therefore we set

$$\hat{\kappa}_{12}=\kappa^{\pm}_{12}\hat{\varrho}^{(c)}_{0\pm}|\mathring{1}\rangle\langle\mathring{2}|\,\hat{\varrho}^{(c)T}_{0\pm}, \tag{7.8}$$

$$\hat{\kappa}_{21}=\kappa^{\pm}_{21}(\hat{\varrho}^{(c)T}_{0\pm})^{-1}|\mathring{2}\rangle\langle\mathring{1}|\,(\hat{\varrho}^{(c)}_{0\pm})^{-1}. \tag{7.9}$$

Here $\kappa^{\pm}_{12}$ and $\kappa^{\pm}_{21}$ are constants, which should be defined. One should choose in both formulæ the sign plus for sectors I, II, VI, but the sign minus for sectors III, IV, V. However, operators $\hat{\kappa}_{jk}$, being the coefficients of the asymptotic expansion of the operator $\hat{U}$, must be the same in any sector of complex ξ-plane (since, by definition, they do not depend on ξ). Therefore, the values of κ^{+}_{jk} and κ^{-}_{jk} on the contour $C_{II-III}\cup C_{V-VI}$ should be related with the jump of the operator $\hat{\varrho}^{(c)}_{0}$ on the same contour, in order to provide the unique value of the operators $\hat{\kappa}_{jk}$. Thus, we demand

$$\kappa^{+}_{12}\hat{\varrho}^{(c)}_{0+}|\mathring{1}\rangle\langle\mathring{2}|\,\hat{\varrho}^{(c)T}_{0+}=\kappa^{-}_{12}\hat{\varrho}^{(c)}_{0-}|\mathring{1}\rangle\langle\mathring{2}|\,\hat{\varrho}^{(c)T}_{0-}. \tag{7.10}$$

Using (6.13)–(6.15) we find

$$(\hat{\varrho}^{(c)}_{0-})^{-1}\hat{\varrho}^{(c)}_{0+}=\left(\hat{\imath}+f_2(\lambda_0)|\mathring{1}\rangle\langle\mathring{1}|\right)^{-1},$$
$$\left((\hat{\varrho}^{(c)}_{0+})^{-1}\hat{\varrho}^{(c)}_{0-}\right)^{T}=\hat{\imath}+f_2(\lambda_0)|\mathring{2}\rangle\langle\mathring{2}|\,. \tag{7.11}$$

This leads us to the jump condition for κ_{12}

$$\kappa^{-}_{12}=\kappa^{+}_{12}(1+f_2(\lambda_0))^{-2}. \tag{7.12}$$

Similar consideration gives

$$\kappa^{-}_{21}=\kappa^{+}_{21}(1+f_2(\lambda_0))^{2}. \tag{7.13}$$

Substituting (7.8), (7.9) into expression (6.47) for $\hat{\nu}$ we obtain

$$\hat{\nu}=\nu\hat{\varrho}^{(c)}_{0}|\mathring{1}\rangle\langle\mathring{1}|\,(\hat{\varrho}^{(c)}_{0})^{-1}, \tag{7.14}$$

and

$$\hat{\tilde{\nu}}=-\nu(\hat{\varrho}^{(c)T}_{0})^{-1}|\mathring{2}\rangle\langle\mathring{2}|\,\hat{\varrho}^{(c)T}_{0}=-\hat{\nu}^{T}, \tag{7.15}$$

where

$$\nu=-2it\kappa^{+}_{12}\kappa^{+}_{21}=-2it\kappa^{-}_{12}\kappa^{-}_{21}. \tag{7.16}$$

It is worth mentioned that due to (7.12) and (7.13) the number ν (not only the operator $\hat{\nu}$!) is the same in all sectors of ξ-plane.

Using explicit expressions for the operators $\hat{\nu}$ and $\hat{\hat{\nu}}$ (7.14), (7.15) and formulæ (7.5), (7.7) one can define 'operator-indexed' PCFs as

$$D_{\hat{\nu}}(\xi) = \hat{\varrho}_0^{(c)} \left[D_0(\xi) \left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}| \right) + |\mathring{1}\rangle\langle\mathring{1}|\, D_\nu(\xi) \right] (\hat{\varrho}_0^{(c)})^{-1}. \tag{7.17}$$

$$D_{\hat{\nu}-\hat{\imath}}(\xi) = \hat{\varrho}_0^{(c)} \left[D_{-1}(\xi) \left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}| \right) + |\mathring{1}\rangle\langle\mathring{1}|\, D_{\nu-1}(\xi) \right] (\hat{\varrho}_0^{(c)})^{-1}. \tag{7.18}$$

$$D_{\hat{\hat{\nu}}}(i\xi) = (\hat{\varrho}_0^{(c)T})^{-1} \left[D_0(i\xi) \left(\hat{\imath} - |\mathring{2}\rangle\langle\mathring{2}| \right) + |\mathring{2}\rangle\langle\mathring{2}|\, D_{-\nu}(i\xi) \right] \hat{\varrho}_0^{(c)T}. \tag{7.19}$$

$$D_{\hat{\hat{\nu}}-\hat{\imath}}(i\xi) = (\hat{\varrho}_0^{(c)T})^{-1} \left[D_{-1}(i\xi) \left(\hat{\imath} - |\mathring{2}\rangle\langle\mathring{2}| \right) + |\mathring{2}\rangle\langle\mathring{2}|\, D_{-\nu-1}(i\xi) \right] \hat{\varrho}_0^{(c)T}. \tag{7.20}$$

We would like to mention that ν in these formulas is a number, and therefore $D_\nu(\xi)$ is the usual (not operator-indexed!) PCF. It follows from (7.11) that

$$\hat{\varrho}_{0-}^{(c)} |\mathring{1}\rangle\langle\mathring{1}|\, (\hat{\varrho}_{0-}^{(c)})^{-1} = \hat{\varrho}_{0+}^{(c)} |\mathring{1}\rangle\langle\mathring{1}|\, (\hat{\varrho}_{0+}^{(c)})^{-1}, \tag{7.21}$$

and

$$(\hat{\varrho}_{0-}^{(c)T})^{-1} |\mathring{2}\rangle\langle\mathring{2}|\, \hat{\varrho}_{0-}^{(c)T} = (\hat{\varrho}_{0+}^{(c)T})^{-1} |\mathring{2}\rangle\langle\mathring{2}|\, \hat{\varrho}_{0+}^{(c)T}. \tag{7.22}$$

Therefore, the r.h.s.s of (7.17)- (7.20) do not have jumps on the boundaries C_{II-III} and C_{V-VI}.

One can check once more, that functions, defined by (7.17)- (7.20) satisfy recurrence (6.45).

8 Reduction to the matrix RHP

Define two mappings $\hat{\mathcal{A}}_0^{\|}(M)$ ('parallel mapping') and $\hat{\mathcal{A}}_0^{\perp}(M)$ ('orthogonal mapping') of 2×2 matrices into algebra of 'hat'-operators. The first mapping is defined for arbitrary matrix M:

$$\hat{\mathcal{A}}_0^{\|}(M) = \begin{pmatrix} M_{11} |\mathring{1}\rangle\langle\mathring{1}| & M_{12} |\mathring{1}\rangle\langle\mathring{2}| \\ M_{21} |\mathring{2}\rangle\langle\mathring{1}| & M_{22} |\mathring{2}\rangle\langle\mathring{2}| \end{pmatrix}, \tag{8.1}$$

The second mapping is defined only for diagonal 2×2 matrices:

$$\hat{\mathcal{A}}_0^{\perp}(M) = \begin{pmatrix} \left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}| \right) M_{11} & 0 \\ 0 & \left(\hat{\imath} - |\mathring{2}\rangle\langle\mathring{2}| \right) M_{22} \end{pmatrix}. \tag{8.2}$$

Similar mappings were introduced by Korepin, and they have already been used in [47]. Similar to [47] it is strightforward to check that

$$M \mapsto \hat{\mathcal{A}}_0(M) \equiv \hat{\mathcal{A}}_0^{\perp}(I) + \hat{\mathcal{A}}_0^{\parallel}(M), \tag{8.3}$$

where I is the unite matrix, is a representation of $Gl(2,C)$ (cf. (1.19), (1.21) in the introduction) . We would like to emphasize that in distinction of the representation, considered in [47], *the mappings* $\hat{\mathcal{A}}_0^{\parallel}(M)$ *and* $\hat{\mathcal{A}}_0^{\perp}(M)$ *preserve the analytical structure of a matrix function* $M(\lambda)$ since the projectors $|\mathring{1}\rangle\langle\mathring{1}|$, $|\mathring{1}\rangle\langle\mathring{2}|$ etc. do not depend on λ.

The mappings $\hat{\mathcal{A}}_0^{\parallel}(M)$ and $\hat{\mathcal{A}}_0^{\perp}(M)$ are not representations. In particular $\hat{\mathcal{A}}_0^{\parallel}(I) \neq \hat{I}$ and $\hat{\mathcal{A}}_0^{\perp}(I) \neq \hat{I}$. Nevertheless these mappings possess very important properties:

$$\begin{aligned} &\hat{\mathcal{A}}_0^{\parallel}(M)\hat{\mathcal{A}}_0^{\perp}(N) = \hat{\mathcal{A}}_0^{\perp}(N)\hat{\mathcal{A}}_0^{\parallel}(M) = 0, \\ &\hat{\mathcal{A}}_0^{\parallel}(M)\hat{\mathcal{A}}_0^{\parallel}(N) = \hat{\mathcal{A}}_0^{\parallel}(MN), \\ &\hat{\mathcal{A}}_0^{\perp}(M)\hat{\mathcal{A}}_0^{\perp}(N) = \hat{\mathcal{A}}_0^{\perp}(MN). \end{aligned} \tag{8.4}$$

These properties allows one to reduce the operator-valued RHP (i) for $\hat{\Psi}$ to two matrix RHP for parallel and orthogonal parts of its pre-image.

Let

$$\hat{\mathcal{R}}^{(0)} = \begin{pmatrix} \hat{\varrho}_0^{(c)} & 0 \\ 0 & (\hat{\varrho}_0^{(c)T})^{-1} \end{pmatrix}. \tag{8.5}$$

Since $\hat{\varrho}_0^{(c)}$ has two values $\hat{\varrho}_{0+}^{(c)}$ and $\hat{\varrho}_{0-}^{(c)}$, the operator-valued matrix $\hat{\mathcal{R}}^{(0)}$ also has two values:$\hat{\mathcal{R}}_+^{(0)}$ in sectors I, II, VI, and $\hat{\mathcal{R}}_-^{(0)}$ in sectors III, IV, V. The jump condition is

$$\hat{\mathcal{R}}_-^{(0)} = \hat{\mathcal{R}}_+^{(0)}\hat{K}_-, \tag{8.6}$$

where $\hat{K}_-$ is defined in (6.35).

Let us introduce also two scalar matrices $\beta^{\perp}$ and $\beta^{\parallel}$:

$$\beta^{\perp} = \begin{pmatrix} D_0(\xi) & 0 \\ 0 & D_0(i\xi) \end{pmatrix}, \tag{8.7}$$

$$\beta^{\parallel} = \begin{pmatrix} D_\nu(\xi) & \kappa_{12}\sqrt{2t}e^{i\frac{\pi}{4}}D_{-\nu-1}(i\xi) \\ \kappa_{21}\sqrt{2t}e^{-i\frac{\pi}{4}}D_{\nu-1}(\xi) & D_{-\nu}(i\xi) \end{pmatrix}. \tag{8.8}$$

The matrix $\beta^{\parallel}$ depends on κ_{12} and κ_{21}, which have two values in different sectors of the complex plane. Hence, $\beta^{\parallel}$ also has two values: $\beta_+^{\parallel}$ in sectors I, II, VI, and $\beta_-^{\parallel}$ in sectors III, IV, V. The jump condition is (cf. (7.12), (7.13))

$$\beta_-^{\parallel} = (k_-)^{-1}\beta_+^{\parallel} \cdot k_-, \tag{8.9}$$

where

$$k_{-} = \begin{pmatrix} 1+f_2(\lambda_0) & 0 \\ 0 & (1+f_2(\lambda_0))^{-1} \end{pmatrix}. \tag{8.10}$$

Let matrix-operator $\hat{L}$ has the structure

$$\hat{L} = \hat{\mathcal{R}}^{(0)} \left(\hat{\mathcal{A}}_0^{\perp}(\ell^{\perp}) + \hat{\mathcal{A}}_0^{\parallel}(\ell^{\parallel}) \right), \tag{8.11}$$

where

$$\ell^{\perp} = \begin{pmatrix} \ell_{11}^{\perp} & 0 \\ 0 & \ell_{22}^{\perp} \end{pmatrix}, \tag{8.12}$$

and

$$\ell^{\parallel} = \begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix}. \tag{8.13}$$

Then combining Eqs. (7.17)-(7.20), (8.7)–(8.13) and using the properties of mappings (8.4) we obtain

$$\hat{\Psi} = \hat{\mathcal{R}}^{(0)} \Big(\hat{\mathcal{A}}_0^{\perp}(\beta^{\perp}\ell^{\perp}) + \hat{\mathcal{A}}_0^{\parallel}(\beta^{\parallel}\ell^{\parallel}) \Big). \tag{8.14}$$

It is easy to check, that the matrix $\hat{S}^{-1}$ (see (6.36)) also has the structure:

$$\hat{S}^{-1} = \hat{\mathcal{R}}^{(0)} \left(\hat{\mathcal{A}}_0^{\perp}(s^{\perp}) + \hat{\mathcal{A}}_0^{\parallel}(s^{\parallel}) \right), \tag{8.15}$$

where

$$s^{\perp} = \begin{pmatrix} e^{-\frac{\xi^2}{4} - \frac{it\lambda_0^2}{2}} & 0 \\ 0 & e^{\frac{\xi^2}{4} + \frac{it\lambda_0^2}{2}} \end{pmatrix}, \tag{8.16}$$

$$s^{\parallel} = \begin{pmatrix} e^{-\frac{\xi^2}{4}} \xi^{is} (2t)^{-\frac{is}{2}} e^{-\frac{\pi s}{4} - \frac{it\lambda_0^2}{2}} & 0 \\ 0 & e^{\frac{\xi^2}{4}} \xi^{-is} (2t)^{\frac{is}{2}} e^{\frac{\pi s}{4} + \frac{it\lambda_0^2}{2}} \end{pmatrix}. \tag{8.17}$$

Here we have used formulæ (7.4) for $\hat{\omega}_0$ and $\hat{\omega}_0^T$.

Consider now the mappings for jump matrices. The jump conditions for the operator $\hat{\Psi}$ at the contours C_{I-VI}, C_{I-II}, C_{III-IV} and C_{IV-V} are given by the matrix $\hat{\mathcal{W}}_{\Psi} = \{\hat{N}_{\pm,\Psi}^{(0)}, \hat{M}_{\pm,\Psi}^{(0)}\}$ (see Eqs.(6.28)–(6.32) and Fig.4). Formulæ for $\hat{Q}^{(0)}$, $\hat{P}^{(0)}$, $\hat{\tilde{Q}}^{(0)}$ and $\hat{\tilde{P}}^{(0)}$ are given in (6.23)–(6.24). Using explicit expressions for $\hat{G}_{jk}$ (see (3.11)) we find

$$\hat{Q}^{(0)} = q^{(0)} |\mathring{2}\rangle\langle\mathring{1}| , \qquad \hat{\tilde{Q}}^{(0)} = \tilde{q}^{(0)} |\mathring{2}\rangle\langle\mathring{1}| , \tag{8.18}$$

$$\hat{P}^{(0)} = p^{(0)} |\mathring{1}\rangle\langle\mathring{2}| , \qquad \hat{\tilde{P}}^{(0)} = \tilde{p}^{(0)} |\mathring{1}\rangle\langle\mathring{2}| , \tag{8.19}$$

where

$$q^{(0)} = \frac{Z_0\vartheta_0 e^{\phi_D(\lambda_0)+\phi_A(\lambda_0)-\psi(\lambda_0)}}{2\pi i(1-Z_0\vartheta_0 e^{\phi_D(\lambda_0)})}, \qquad p^{(0)} = \frac{2\pi i(\vartheta_0-1)Z_0 e^{\psi(\lambda_0)}}{1-Z_0\vartheta_0 e^{\phi_D(\lambda_0)}}, \tag{8.20}$$

$$\tilde{q}^{(0)} = \frac{Z_0\vartheta_0 e^{\phi_D(\lambda_0)+\phi_A(\lambda_0)-\psi(\lambda_0)}}{2\pi i(1-Z_0\vartheta_0 e^{\phi_A(\lambda_0)})}, \qquad \tilde{p}^{(0)} = \frac{2\pi i(\vartheta_0-1)Z_0 e^{\psi(\lambda_0)}}{1-Z_0\vartheta_0 e^{\phi_A(\lambda_0)}}. \tag{8.21}$$

Obviously, all the jump matrices $\hat{\mathcal{W}}_\Psi$ can be written in the form

$$\hat{\mathcal{W}}_\Psi = \hat{\mathcal{A}}_0^{\perp}(I) + \hat{\mathcal{A}}_0^{\parallel}(w_\Psi). \tag{8.22}$$

Here

$$w_\Psi = \{n_\pm, m_\pm\}, \tag{8.23}$$

and

$$m_+ = \begin{pmatrix} 1 & 0 \\ q^{(0)} & 1 \end{pmatrix}, \qquad n_- = \begin{pmatrix} 1 & 0 \\ \tilde{q}^{(0)} & 1 \end{pmatrix}, \tag{8.24}$$

$$m_- = \begin{pmatrix} 1 & p^{(0)} \\ 0 & 1 \end{pmatrix}, \qquad n_+ = \begin{pmatrix} 1 & \tilde{p}^{(0)} \\ 0 & 1 \end{pmatrix}. \tag{8.25}$$

The jump condition at the contours C_{II-III} and C_{V-VI} are given by matrix $\hat{K}_\pm$, which also can be presented as

$$\hat{K}_\pm = \hat{\mathcal{A}}_0^{\perp}(I) + \hat{\mathcal{A}}_0^{\parallel}(k_\pm), \tag{8.26}$$

where

$$k_+ = \begin{pmatrix} 1+f_1(\lambda_0) & 0 \\ 0 & (1+f_1(\lambda_0))^{-1} \end{pmatrix}, \tag{8.27}$$

$$k_- = \begin{pmatrix} 1+f_2(\lambda_0) & 0 \\ 0 & (1+f_2(\lambda_0))^{-1} \end{pmatrix}. \tag{8.28}$$

The jump matrices $m_\pm$, $n_\pm$ and $k_\pm$ are placed on the Fig.4 in the parenthesis.

Thus all operators entering the RHP (i) and its solution are written as the $\hat{\mathcal{A}}_0()$- mappings of 2×2 matrices. Let us summarize the preliminary results.

Using the mappings $\hat{\mathcal{A}}_0^{\perp}(\cdot)$ and $\hat{\mathcal{A}}_0^{\parallel}(\cdot)$ we have presented the operator-valued matrix $\hat{\Psi}$ as a sum of orthogonal and parallel parts (see (8.14)):

$$\hat{\Psi} = \hat{\mathcal{R}}^{(0)}\Big(\hat{\mathcal{A}}_0^{\perp}(\beta^{\perp}\ell^{\perp}) + \hat{\mathcal{A}}_0^{\parallel}(\beta^{\parallel}\ell^{\parallel})\Big). \tag{8.29}$$

We have similar representation for the operator $\hat{S}^{-1}$

$$\hat{S}^{-1} = \hat{\mathcal{R}}^{(0)}\left(\hat{\mathcal{A}}_0^{\perp}(s^{\perp}) + \hat{\mathcal{A}}_0^{\parallel}(s^{\parallel})\right). \tag{8.30}$$

This gives us the asymptotic conditions for orthogonal and parallel parts of $\hat{\Psi}$. Indeed, since $\hat{\Psi} \to \hat{S}^{-1}$, when ξ goes to infinity, then the pre-images of the parallel and orthogonal parts of $\hat{\Psi}$ should go to the pre-images of the parallel and orthogonal parts of $\hat{S}^{-1}$ respectively:

$$\beta^{\perp}\ell^{\perp} \overset{\xi\to\infty}{\longrightarrow} s^{\perp}, \tag{8.31}$$

$$\beta^{\parallel}\ell^{\parallel} \overset{\xi\to\infty}{\longrightarrow} \begin{pmatrix} s_{11}^{\parallel} & \kappa_{12}\sqrt{2t}e^{-i\frac{\pi}{4}}\xi^{-1}s_{22}^{\parallel} \\ \kappa_{21}\sqrt{2t}e^{-i\frac{\pi}{4}}\xi^{-1}s_{11}^{\parallel} & s_{22}^{\parallel} \end{pmatrix}. \tag{8.32}$$

Finally, all the jump matrices also are written in terms of orthogonal and parallel parts:

$$\hat{\mathcal{W}}_{\Psi} = \hat{\mathcal{A}}_0^{\perp}(I) + \hat{\mathcal{A}}_0^{\parallel}(w_{\Psi}), \tag{8.33}$$

$$\hat{K}_{\pm} = \hat{\mathcal{A}}_0^{\perp}(I) + \hat{\mathcal{A}}_0^{\parallel}(k_{\pm}). \tag{8.34}$$

The parallel and orthogonal parts of $\hat{\Psi}$ should separately satisfy jump conditions. Thus, we have reduced the operator-valued RHP to two matrix ones. Namely, we need now to find piecewise constant matrices $\ell^{\perp}$, $\ell^{\parallel}$ and constants $\kappa_{12}^{\pm}$ and $\kappa_{21}^{\pm}$ such that:

1. both of the matrices $\beta^{\perp}\ell^{\perp}$ and $\beta^{\parallel}\ell^{\parallel}$ possess the asymptotics prescribed;
2. both of the matrices $\beta^{\perp}\ell^{\perp}$ and $\beta^{\parallel}\ell^{\parallel}$ possess the jump conditions prescribed.

We start with the asymptotic condition for the orthogonal part. One should find matrix $\ell^{\perp}$ such that

$$\beta^{\perp}\ell^{\perp} \overset{\xi\to\infty}{\longrightarrow} \begin{pmatrix} e^{-\frac{\xi^2}{4}-\frac{it\lambda_0^2}{2}} & 0 \\ 0 & e^{\frac{\xi^2}{4}+\frac{it\lambda_0^2}{2}} \end{pmatrix}. \tag{8.35}$$

Here $\beta^{\perp}$ is given by (8.7):

$$\beta^{\perp} = \begin{pmatrix} D_0(\xi) & 0 \\ 0 & D_0(i\xi) \end{pmatrix}. \tag{8.36}$$

Since $D_0(\xi) = \exp(-\xi^2/4)$, we conclude that

$$\ell^{\perp} = \exp\{-\frac{it\lambda_0^2}{2}\sigma_3\}. \tag{8.37}$$

Let us discuss now the jump conditions for orthogonal part. Observe that orthogonal parts of all jump matrices are equal to $\hat{\mathcal{A}}_0^{\perp}(I)$. Thus, matrix $\beta^{\perp}\ell^{\perp}$ has no jumps on the contours C_{I-VI}, C_{I-II}, C_{III-IV} and C_{IV-V}. It also has no jumps at the contours C_{II-III} and C_{V-VI} in spite of orthogonal part of $\hat{\Psi}$ depends on matrix $\hat{\mathcal{R}}^{(0)}$, which may have jump just at the contours C_{II-III} and C_{V-VI}. Let us check the corresponding jump condition. We have (see (7.11))

$$\hat{\varrho}_{0-}^{(c)} = \hat{\varrho}_{0+}^{(c)}\left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}| + (1+f_2(\lambda_0))|\mathring{1}\rangle\langle\mathring{1}|\right), \tag{8.38}$$

(see (6.15)). Multiplying this equality by $\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}|$ from the right, we obtain

$$\hat{\varrho}_{0-}^{(c)}\left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}|\right) = \hat{\varrho}_{0+}^{(c)}\left(\hat{\imath} - |\mathring{1}\rangle\langle\mathring{1}|\right). \tag{8.39}$$

Thus, the orthogonal part of $\hat{\varrho}_0^{(c)}$ (and, hence, orthogonal part of $\hat{\mathcal{R}}^{(0)}$) has no jump at the contours C_{II-III} and C_{V-VI}.

The orthogonal part of $\hat{\Psi}$ is found:

$$\hat{\Psi}^{\perp} = \hat{\mathcal{R}}^{(0)}\hat{\mathcal{A}}_0^{\perp}(\beta^{\perp}\ell^{\perp}), \tag{8.40}$$

where $\beta^{\perp}$ and $\ell^{\perp}$ are given by (8.36) and (8.37).

Consider now the parallel part of $\hat{\Psi}$. In order to simplify some formulæ, we make replacement

$$\ell^{\parallel} = \tilde{\ell}\exp\left(\left\{-\frac{is}{2}\ln(2t) - \frac{\pi s}{4} - \frac{it\lambda_0^2}{2}\right\}\sigma_3\right). \tag{8.41}$$

First, one should find piecewise constant matrix $\tilde{\ell}$ such that matrix $\beta^{\parallel}\tilde{\ell}$ possesses the following asymptotics

$$\beta^{\parallel}\tilde{\ell} \overset{\xi\to\infty}{\longrightarrow} \begin{pmatrix} \xi^{is}e^{-\frac{\xi^2}{4}} & \kappa_{12}\sqrt{2t}e^{-i\frac{\pi}{4}}\xi^{-is-1}e^{\frac{\xi^2}{4}} \\ \kappa_{21}\sqrt{2t}e^{-i\frac{\pi}{4}}\xi^{is-1}e^{-\frac{\xi^2}{4}} & \xi^{-is}e^{\frac{\xi^2}{4}} \end{pmatrix}. \tag{8.42}$$

Recall that $\beta^{\parallel}$ is equal to

$$\beta^{\parallel} = \begin{pmatrix} D_{\nu}(\xi) & \kappa_{12}\sqrt{2t}e^{i\frac{\pi}{4}}D_{-\nu-1}(i\xi) \\ \kappa_{21}\sqrt{2t}e^{-i\frac{\pi}{4}}D_{\nu-1}(\xi) & D_{-\nu}(i\xi) \end{pmatrix}. \tag{8.43}$$

The asymptotics of PCF $D_{\nu}(\xi)$ strongly depends on argument of ξ. In general form it may be written as

$$D_{\nu}(\xi) \overset{|\xi|\to\infty}{\longrightarrow} \xi^{\nu}e^{-\frac{\xi^2}{4}} + c(\nu)\xi^{-\nu-1}e^{\frac{\xi^2}{4}}, \tag{8.44}$$

where constant $c(\nu)$ does not depends on $|\xi|$, but only on $\arg\xi$ [49]. It easy to see that the asymptotics of (8.42) can coincide with (8.43) only if

$$\nu = is. \tag{8.45}$$

Then we need to choose matrix $\tilde{\ell}$ in order to provide the correct asymptotics behavior in every sectors of the complex ξ-plane. We drop out the detail of these calculations and present here the matrix $\tilde{\ell}$ in each sector:

$$\tilde{\ell} = \begin{pmatrix} 1 & \sqrt{\frac{\pi}{t}}\frac{e^{-\frac{i\pi}{4}}}{\kappa_{21}\Gamma(\nu)} \\ 0 & e^{\frac{i\pi\nu}{2}} \end{pmatrix}, \qquad \text{in the sector } I, \tag{8.46}$$

$$\tilde{\ell} = \begin{pmatrix} e^{2i\pi\nu} & \sqrt{\dfrac{\pi}{t}} \dfrac{e^{-\frac{i\pi}{4}}}{\kappa_{21}\Gamma(\nu)} \\ \sqrt{\dfrac{\pi}{t}} \dfrac{e^{\frac{3i\pi\nu}{2}+\frac{i\pi}{4}}}{\kappa_{12}\Gamma(-\nu)} & e^{\frac{i\pi\nu}{2}} \end{pmatrix}, \qquad \text{in the sector } II, \tag{8.47}$$

$$\tilde{\ell} = \begin{pmatrix} 1 & \sqrt{\dfrac{\pi}{t}} \dfrac{e^{2i\pi\nu-\frac{i\pi}{4}}}{\kappa_{21}\Gamma(\nu)} \\ \sqrt{\dfrac{\pi}{t}} \dfrac{e^{-\frac{i\pi\nu}{2}+\frac{i\pi}{4}}}{\kappa_{12}\Gamma(-\nu)} & e^{\frac{5i\pi\nu}{2}} \end{pmatrix}, \qquad \text{in the sector } III, \tag{8.48}$$

$$\tilde{\ell} = \begin{pmatrix} 1 & 0 \\ \sqrt{\dfrac{\pi}{t}} \dfrac{e^{-\frac{i\pi\nu}{2}+\frac{i\pi}{4}}}{\kappa_{12}\Gamma(-\nu)} & e^{\frac{i\pi\nu}{2}} \end{pmatrix}, \qquad \text{in the sector } IV, \tag{8.49}$$

$$\tilde{\ell} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi\nu}{2}} \end{pmatrix}, \qquad \text{in the sectors } V \text{ and } VI. \tag{8.50}$$

Let us illustrate how one can obtain these results. Consider sectors V and VI where $-\frac{\pi}{2} \leq \arg\xi \leq 0$. For these values of argument the asymptotics of the PCF is extremely simple:

$$D_\nu(\xi) \to \xi^\nu e^{-\frac{\xi^2}{4}}, \tag{8.51}$$

$$D_{-\nu}(i\xi) \to e^{-\frac{i\pi\nu}{2}} \xi^{-\nu} e^{\frac{\xi^2}{4}}. \tag{8.52}$$

Putting $\tilde{\ell}_{12} = \tilde{\ell}_{21} = 0$, $\tilde{\ell}_{11} = 1$ and $\tilde{\ell}_{22} = e^{\frac{i\pi\nu}{2}}$ we satisfy the asymptotic condition. It is worth mentioned that matrix $\tilde{\ell}$ has no jump at the contour C_{V-VI}, which is in completely agreement with the jump condition (8.65) below. Thus, we arrive at (8.50).

Consider now sector I. Here the asymptotics of $D_\nu(\xi)$ is just the same, but $D_{-\nu}(i\xi)$ behaves as follows

$$D_{-\nu}(i\xi) \to e^{-\frac{i\pi\nu}{2}} \left(\xi^{-\nu} e^{\frac{\xi^2}{4}} - \frac{\sqrt{2\pi}}{\Gamma(\nu)} e^{-i\frac{\pi}{2}} \xi^{\nu-1} e^{-\frac{\xi^2}{4}} \right). \tag{8.53}$$

Putting again $\tilde{\ell}_{21} = 0$ and $\tilde{\ell}_{11} = 1$, we obtain, that the first column of $\beta^{\parallel}\tilde{\ell}$ possesses correct asymptotics. But the second column has an additional term in comparison with sectors V and VI. In order to compensate this term, one should put $\tilde{\ell}_{22} = e^{\frac{i\pi\nu}{2}}$ and $\tilde{\ell}_{12} = \sqrt{\frac{\pi}{t}} \frac{e^{-\frac{i\pi}{4}}}{\kappa_{21}\Gamma(\nu)}$. We obtain formula (8.46).

Similar considerations allow to find all other formulæ (8.47)–(8.49) for matrix $\tilde{\ell}$.

Finally, we should check, that all the jump conditions are valid. After replacement $\ell^{\parallel} \to \tilde{\ell}$ the jump conditions for $\beta^{\parallel}\tilde{\ell}$ have changed. Let us formulate them explicitly.

The diagonal matrices $k_{\pm}$ do not change under replacement (8.41). However, the matrices $m_{\pm}$ and $n_{\pm}$ should be replaced by $m'_{\pm}$ and $n'_{\pm}$ respectively:

$$m'_{+} = \begin{pmatrix} 1 & 0 \\ q'^{(0)} & 1 \end{pmatrix}, \qquad n'_{-} = \begin{pmatrix} 1 & 0 \\ \tilde{q}'^{(0)} & 1 \end{pmatrix}, \tag{8.54}$$

$$m'_{-} = \begin{pmatrix} 1 & p'^{(0)} \\ 0 & 1 \end{pmatrix}, \qquad n'_{+} = \begin{pmatrix} 1 & \tilde{p}'^{(0)} \\ 0 & 1 \end{pmatrix}, \tag{8.55}$$

where

$$q'^{(0)} = q^{(0)}(2t)^{\nu}e^{-\frac{i\pi\nu}{2}+it\lambda_0^2}, \qquad \tilde{q}'^{(0)} = \tilde{q}^{(0)}(2t)^{\nu}e^{-\frac{i\pi\nu}{2}+it\lambda_0^2}, \tag{8.56}$$

$$p'^{(0)} = p^{(0)}(2t)^{-\nu}e^{\frac{i\pi\nu}{2}-it\lambda_0^2}, \qquad \tilde{p}'^{(0)} = \tilde{p}^{(0)}(2t)^{-\nu}e^{\frac{i\pi\nu}{2}-it\lambda_0^2}. \tag{8.57}$$

Thus, the matrix $\tilde{\ell}$ should possess the following jumps:

$$\tilde{\ell}_{VI} = \tilde{\ell}_{I} n'_{+}, \qquad \text{at the contour } C_{I-VI}, \tag{8.58}$$

$$\tilde{\ell}_{II} = \tilde{\ell}_{I} m'_{+}, \qquad \text{at the contour } C_{I-II}, \tag{8.59}$$

$$\tilde{\ell}_{IV} = \tilde{\ell}_{III} m'_{-}, \qquad \text{at the contour } C_{III-IV}, \tag{8.60}$$

$$\tilde{\ell}_{IV} = \tilde{\ell}_{V} n'_{-}, \qquad \text{at the contour } C_{IV-V}. \tag{8.61}$$

As for contours C_{II-III} and C_{V-VI}, the situation is slightly more complicated. Recall jump conditions for $\hat{\mathcal{R}}^{(0)}$ and $\beta^{\parallel}$

$$\hat{\mathcal{R}}^{(0)}_{-} = \hat{\mathcal{R}}^{(0)}_{+}(\hat{\mathcal{A}}_0^{\perp}(I) + \hat{\mathcal{A}}_0^{\parallel}(k_{-})), \tag{8.62}$$

$$\beta^{\parallel}_{-} = (k_{-})^{-1}\beta^{\parallel}_{+} \cdot k_{-}. \tag{8.63}$$

Then for the parallel part of $\hat{\Psi}$ we have at the contour C_{V-VI}:

$$\begin{aligned} \hat{\Psi}^{\parallel}_{-} &= \hat{\mathcal{R}}^{(0)}_{-}\hat{\mathcal{A}}_0^{\parallel}(\beta^{\parallel}_{-}\tilde{\ell}_{V}) = \hat{\mathcal{R}}^{(0)}_{+}\hat{\mathcal{A}}_0^{\parallel}(\beta^{\parallel}_{+}k_{-}\tilde{\ell}_{V}(k_{-})^{-1})\hat{\mathcal{A}}_0^{\parallel}(k_{-}) \\ &= \hat{\mathcal{R}}^{(0)}_{+}\hat{\mathcal{A}}_0^{\parallel}(\beta^{\parallel}_{+}\tilde{\ell}_{VI})\hat{\mathcal{A}}_0^{\parallel}(k_{-}). \end{aligned} \tag{8.64}$$

Hence, we find that

$$\tilde{\ell}_{VI} = k_{-}\tilde{\ell}_{V}(k_{-})^{-1}, \qquad \text{at the contour } C_{V-VI}. \tag{8.65}$$

Similarly

$$\tilde{\ell}_{II} = k_{-}\tilde{\ell}_{III}(k_{+})^{-1}, \qquad \text{at the contour } C_{II-III}. \tag{8.66}$$

One can easily check that the jumps on the contours C_{V-VI} and C_{II-III} are valid automatically. The jumps on the other contours give us explicit expressions for $\kappa_{12}^{\pm}$ and $\kappa_{21}^{\pm}$. Indeed, for example, due to (8.58) we have

$$(\tilde{\ell}_{VI})_{12} = (\tilde{\ell}_I)_{12} + \tilde{p}'^{(0)}(\tilde{\ell}_I)_{11}, \tag{8.67}$$

which implies

$$(\tilde{\ell}_I)_{12} = -\tilde{p}'^{(0)}, \tag{8.68}$$

and, hence,

$$\kappa_{21}^{+} = -\sqrt{\frac{\pi}{t}}\frac{e^{-\frac{i\pi}{4}}}{\tilde{p}'^{(0)}\Gamma(\nu)}. \tag{8.69}$$

Then due to (7.16)

$$\kappa_{12}^{+} = \frac{i\nu}{2t\kappa_{21}^{+}}. \tag{8.70}$$

Using equalities (7.12) and (7.13) we can find κ_{12}^{-} and κ_{21}^{-}.

The jumps on the other contours give us just the same expressions for $\kappa_{jk}^{\pm}$. One should only take into account the identity

$$\tilde{p}'^{(0)}q'^{(0)} = \tilde{q}'^{(0)}p'^{(0)} = 1 - e^{2i\pi\nu}. \tag{8.71}$$

This identity can be checked directly, using explicit expressions for all variables entering (8.71) (see (8.56)-(8.57), (8.20)-(8.21)).

Combining all necessary formulæ after simple algebra we arrive at

$$\hat{\kappa}_{12} = \frac{\gamma\nu}{2\pi i}\sqrt{\frac{\pi}{t}}\hat{\varrho}_{0-}^{(c)}|\mathring{1}\rangle\langle\mathring{2}|\,\hat{\varrho}_{0+}^{(c)T}, \tag{8.72}$$

$$\hat{\kappa}_{21} = -\frac{1}{\gamma}\sqrt{\frac{\pi}{t}}(\hat{\varrho}_{0+}^{(c)T})^{-1}|\mathring{2}\rangle\langle\mathring{1}|\,(\hat{\varrho}_{0-}^{(c)})^{-1}. \tag{8.73}$$

Here

$$\gamma = 2\pi Z_0(\vartheta_0 - 1)\Gamma(\nu)(2t)^{-\nu}e^{\psi(\lambda_0)+\frac{i\pi\nu}{2}+\frac{3i\pi}{4}-it\lambda_0^2}, \tag{8.74}$$

and

$$\nu = -\frac{1}{2\pi i}\ln\left[\left(1-\vartheta_0 Z_0 e^{\phi_D(\lambda_0)}\right)\left(1-\vartheta_0 Z_0 e^{\phi_A(\lambda_0)}\right)\right]. \tag{8.75}$$

This gives us the solution $\hat{\Psi}$ of the localized RHP (i). Let us present here once more the main formulæ. The operator $\hat{\Psi}$ is equal to

$$\hat{\Psi} = \begin{pmatrix} \hat{\varrho}_0^{(c)} & 0 \\ 0 & (\hat{\varrho}_0^{(c)T})^{-1} \end{pmatrix}\Big(\hat{\mathcal{A}}_0^{\perp}(\beta^{\perp}\ell^{\perp}) + \hat{\mathcal{A}}_0^{\parallel}(\beta^{\parallel}\ell^{\parallel})\Big). \tag{8.76}$$

The definition of the mappings $\hat{\mathcal{A}}_0^{\perp}(\cdot)$ and $\hat{\mathcal{A}}_0^{\|}(\cdot)$ is given in (8.1), (8.2). The matrix $\beta^{\perp}\ell^{\perp}$ is equal to

$$\beta^{\perp}\ell^{\perp} = \exp\left\{-\left(\frac{\xi^2}{4} + \frac{it\lambda_0^2}{2}\right)\sigma_3\right\}. \tag{8.77}$$

The parallel part of the matrix β is

$$\beta^{\|} = \begin{pmatrix} D_\nu(\xi) & \kappa_{12}\sqrt{2t}e^{i\frac{\pi}{4}}D_{-\nu-1}(i\xi) \\ \kappa_{21}\sqrt{2t}e^{-i\frac{\pi}{4}}D_{\nu-1}(\xi) & D_{-\nu}(i\xi) \end{pmatrix}. \tag{8.78}$$

Here $D_\nu(\xi)$ is the PCF, ν is given by (8.75). The constants κ_{12} and κ_{21} have two values: κ_{jk}^{+} for the sectors I, II and VI, and κ_{jk}^{-} for the sectors III, IV and V (see Fig.4). The values κ_{jk}^{+} are defined in (8.69), (8.70). The constants κ_{jk}^{-} are related with the last ones by (7.12), (7.13).

The matrix $\ell^{\|}$ it is given by Eqs. (8.41), (8.46)–(8.50).

For our purposes it is most important to know the operators $\hat{\kappa}_{12}$ and $\hat{\kappa}_{21}$ since the equations,

$$\hat{\kappa}_{12} = \hat{b}_{12}, \qquad \hat{\kappa}_{21} = \hat{b}_{21}. \tag{8.79}$$

These operators are given by (8.72), (8.73).

The case of the positive chemical potential can be delt with in a similar fashion. Instead of the operators $\hat{Q}^{(0)}$ and $\hat{P}^{(0)}$ one should consider operators

$$\hat{Q}_1^{(0)} = \hat{Q}^{(0)}\left(\frac{\lambda_0 - \Lambda_2}{\lambda_0 - \Lambda_1}\right), \qquad \hat{P}_1^{(0)} = \hat{P}^{(0)}\left(\frac{\lambda_0 - \Lambda_1}{\lambda_0 - \Lambda_2}\right). \tag{8.80}$$

The corresponding replacement should be done for the scalars $q^{(0)}$ and $p^{(0)}$ as well. The rest of calculations is just the same as for the case of the negative chemical potential. The difference, however, is that for the negative chemical potential formulæ (8.79) provide us with the leading term of the large time asymptotic expansion of the operators $\hat{b}_{12}$ and $\hat{b}_{21}$, while for the positive chemical potential the operators $\hat{\kappa}_{12}$ and $\hat{\kappa}_{21}$ define only corrections to the coefficients $\hat{b}_{12}$ and $\hat{b}_{21}$. Indeed, in the last case, as we have seen, the leading term of the asymptotic expansion of $\hat{b}_{12}$ is given by (5.43). In comparison with this expression, the operators (8.72), (8.73) have the order of $t^{-1/2}$, hence, they are only corrections to the leading terms. Another words in the case of positive chemical potential the operators $\hat{\kappa}_{12}$ and $\hat{\kappa}_{21}$ are equal to the coefficients of the asymptotic expansion (5.33) $\hat{B}_{12}^{(0)}$ and $\hat{B}_{21}^{(0)}$ respectively:

$$\hat{\kappa}_{12} = \hat{B}_{12}^{(0)}, \qquad \hat{\kappa}_{21} = \hat{B}_{21}^{(0)}. \tag{8.81}$$

Finally, we need to say few words about the operator $\hat{\varrho}_0^{(c)}$, entering Eqs. (8.72), (8.73). This operator is defined by the equation,

$$\hat{\varrho}_0^{(c)} = \hat{\varrho}^{(c)}(\lambda_0) \tag{8.82}$$

where the operator-valued function $\hat{\varrho}^{(c)}(\lambda)$ in turn is defined as a unique solution of the RHP (6.13)-(6.14). This RHP is equivalent to the singular integral equation (cf. (1.15)),

$$\hat{\varrho}^{(c)}_{+}(\lambda) = \hat{\imath} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{d\mu}{\mu - \lambda - i0}\hat{\varrho}^{(c)}_{+}(\mu)\left(\hat{\imath} - \hat{\mathcal{D}}_0(\mu)\right). \tag{8.83}$$

Therefore, to determine $\hat{\varrho}^{(c)}_0$ one has to solve this integral equation. In other words, the time-independent operator factor $\hat{\varrho}^{(c)}_0$ in our final formulae is still a transcendental object. It is clear however, that it does not depend on the function $\psi(\lambda)$ since the jump matrix $\hat{\mathcal{D}}_0(\lambda)$ only depends on the functions $\phi_A(\lambda)$, $\phi_D(\lambda)$, and the ratio $\lambda_0 = x/2t$ (and, of course, on the chemical potential, coupling constant and temperature). We shall see later that this property is sufficient in order to calculate the asymptotics of the correlation function up to the numerical constant (depending on λ_0, h, c, and T).

9 Differential equations

In this section we continue to consider the case of negative chemical potential.

The results of the previous sections provides us the leading terms of the large time asymptotics for the coefficients $\hat{b}_{jk}$. In particular, we have found that

$$\hat{b}_{12} \approx \frac{\tilde{\gamma}\nu}{i\sqrt{2\pi}}(2t)^{-\nu-1/2}e^{\psi(\lambda_0)-it\lambda_0^2}\hat{\varrho}^{(c)}_{0-}|\mathring{1}\rangle\langle\mathring{2}|\,\hat{\varrho}^{(c)T}_{0+}, \tag{9.1}$$

$$\hat{b}_{21} \approx -\frac{\sqrt{2\pi}}{\tilde{\gamma}}(2t)^{\nu-1/2}e^{-\psi(\lambda_0)+it\lambda_0^2}(\hat{\varrho}^{(c)T}_{0+})^{-1}|\mathring{2}\rangle\langle\mathring{1}|\,(\hat{\varrho}^{(c)}_{0-})^{-1}. \tag{9.2}$$

We have extracted explicitly the dependence on t and function $\psi(\lambda)$. The constant $\tilde{\gamma}$ is equal to

$$\tilde{\gamma} = 2\pi Z_0(\vartheta_0 - 1)\Gamma(\nu)e^{\frac{i\pi\nu}{2}+\frac{3i\pi}{4}}, \tag{9.3}$$

and it depends only on functions $\phi_D(\lambda)$ and $\phi_A(\lambda)$. Recall also, that

$$\nu = -\frac{1}{2\pi i}\ln\left[\left(1 - \vartheta_0 Z_0 e^{\phi_D(\lambda_0)}\right)\left(1 - \vartheta_0 Z_0 e^{\phi_A(\lambda_0)}\right)\right]. \tag{9.4}$$

Corrections to the asymptotic formulae (9.1), (9.2) decay faster than $t^{-1/2}$. However, in the poduct $\hat{b}_{12}\hat{b}_{21}$ the oscillations cancel, and after integrating equation (2.20) twice these corrections might produce the non-vanishing contributions into $\det(\tilde{I} + \tilde{V})$.

In order to make our asymptotic estimates more accurate, we use differential equations for $\hat{b}_{12}$ and $\hat{b}_{21}$ (2.21):

$$i\partial_t\hat{b}_{21} + \partial_x^2\hat{b}_{21} = 2\hat{b}_{21}\hat{b}_{12}\hat{b}_{21}, \tag{9.5}$$

$$-i\partial_t\hat{b}_{12} + \partial_x^2\hat{b}_{12} = 2\hat{b}_{12}\hat{b}_{21}\hat{b}_{12}. \tag{9.6}$$

We shall look for the solution of this system in the following form

$$\hat{b}_{12} = (2t)^{-\nu-1/2} e^{-it\lambda_0^2} \hat{\zeta}(\lambda_0, t), \tag{9.7}$$

$$\hat{b}_{21} = (2t)^{\nu-1/2} e^{it\lambda_0^2} \hat{\eta}(\lambda_0, t). \tag{9.8}$$

It is convenient to rewrite the differential equations (9.5), (9.6) in terms of variables λ_0 and t. After simple algebra we arrive at the differential equations for the operators $\hat{\zeta}$, $\hat{\eta}$:

$$-4i\frac{\partial \hat{\zeta}}{\partial t} - \frac{4}{t}(\hat{\zeta}\hat{\eta}\hat{\zeta} - i\nu\hat{\zeta}) = -\frac{1}{t^2}\Big\{\Big[(\nu')^2 \ln^2(2t) - \nu'' \ln(2t)\Big]\hat{\zeta} - 2\nu'\hat{\zeta}' \ln(2t) + \hat{\zeta}''\Big\}, \tag{9.9}$$

$$4i\frac{\partial \hat{\eta}}{\partial t} - \frac{4}{t}(\hat{\eta}\hat{\zeta}\hat{\eta} - i\nu\hat{\eta}) = -\frac{1}{t^2}\Big\{\Big[(\nu')^2 \ln^2(2t) + \nu'' \ln(2t)\Big]\hat{\eta} + 2\nu'\hat{\eta}' \ln(2t) + \hat{\eta}''\Big\}. \tag{9.10}$$

Here prime means the derivative with respect to λ_0. It is easy to see that we can assume the asymptotic expansions [50] obtained for the solutions of the scalar Nonlinear Schrödinger equation:

$$\hat{\zeta} = \hat{\zeta}_0(\lambda_0) + \sum_{n=1}^{\infty}\sum_{k=0}^{2n} \hat{\zeta}_{nk}(\lambda_0)\frac{(\ln(2t))^k}{t^n}, \tag{9.11}$$

$$\hat{\eta} = \hat{\eta}_0(\lambda_0) + \sum_{n=1}^{\infty}\sum_{k=0}^{2n} \hat{\eta}_{nk}(\lambda_0)\frac{(\ln(2t))^k}{t^n}, \tag{9.12}$$

where coefficients $\hat{\zeta}_{nk}$ and $\hat{\eta}_{nk}$ depend on λ_0 only. Substituting these decompositions into (9.9) and (9.10) we obtain for the leading terms ζ_0 and η_0

$$\begin{aligned} i\nu\hat{\zeta}_0 &= \hat{\zeta}_0\hat{\eta}_0\hat{\zeta}_0, \\ i\nu\hat{\eta}_0 &= \hat{\eta}_0\hat{\zeta}_0\hat{\eta}_0, \end{aligned} \tag{9.13}$$

These relations (9.13) are valid automatically, since due to Eqs. (9.1), (9.2) we have

$$\hat{\zeta}_0\hat{\eta}_0 = i\nu \hat{\varrho}_{0-}^{(c)} |\mathring{1}\rangle\langle\mathring{1}| \, (\hat{\varrho}_{0-}^{(c)})^{-1}, \tag{9.14}$$

$$\hat{\eta}_0\hat{\zeta}_0 = i\nu (\hat{\varrho}_{0+}^{(c)T})^{-1} |\mathring{2}\rangle\langle\mathring{2}| \, \hat{\varrho}_{0+}^{(c)T}. \tag{9.15}$$

In turn for the first corrections ζ_{1k} and η_{1k} we obtain the following recurrence

$$\begin{aligned} &\hat{\zeta}_{12}\hat{\eta}_0 + \hat{\zeta}_0\hat{\eta}_{12} = 0, \\ &\hat{\zeta}_{11}\hat{\eta}_0 + \hat{\zeta}_0\hat{\eta}_{11} = \frac{1}{2i}\left(\nu'\hat{\zeta}_0\hat{\eta}_0\right)', \\ &\hat{\zeta}_{10}\hat{\eta}_0 + \hat{\zeta}_0\hat{\eta}_{10} = \frac{1}{2i}\left(\nu'\hat{\zeta}_0\hat{\eta}_0\right)' + \frac{1}{4}\left(\hat{\zeta}_0\hat{\eta}_0' - \hat{\zeta}_0'\hat{\eta}_0\right)'. \end{aligned} \tag{9.16}$$

Similar to the scalar Nonlinear Schrödinger equation the relations (9.16) allow us to find more precise estimate for the product $\hat{b}_{12}\hat{b}_{21} = -i\partial_x\hat{b}_{11}$. Recall that in order to obtain asymptotic

expression of the Fredholm determinant we need to find only trace of $\hat{b}_{11}$, but not $\hat{b}_{11}$ it self. Obviously

$$\operatorname{tr}\hat{\zeta}_0\hat{\eta}_0 = i\nu. \tag{9.17}$$

With the trace of the expression $\left(\hat{\zeta}_0\hat{\eta}_0' - \hat{\zeta}_0'\hat{\eta}_0\right)'$—the situation is more complicated, since we do not know explicit expression for $\hat{\varrho}_0^{(c)}$. On the other hand it is possible to extract the dependence on the function $\psi(\lambda)$. Using formulæ (9.1) and (9.2), we find

$$\hat{\zeta}_0\hat{\eta}_0' - \hat{\zeta}_0'\hat{\eta}_0 = -2i\hat{\nu}\psi'(\lambda_0) + \hat{C}_0, \tag{9.18}$$

where $\hat{C}_0$ does not depend on the function $\psi(\lambda)$. Thus we have

$$\operatorname{tr}\hat{b}_{11}' = -\nu + \frac{i}{2t}(\nu'\nu)'(\ln(2t)+1) - \frac{i}{2t}(\nu\psi'(\lambda_0))' + \frac{1}{t}\operatorname{tr}\hat{C}_0 + \mathcal{O}(\ln^2 t/t^2). \tag{9.19}$$

Note. One could expect that due to the (9.11), (9.12) the last equation is valid up to the terms of order $\ln^4 t/t^2$. These corrections do enter the expressions for $\hat{b}_{12}$ and $\hat{b}_{21}$, however they vanish (as well as the terms $\ln^3 t/t^2$) for the combination $\operatorname{tr}\hat{b}_{12}\hat{b}_{21}$.

Finally one should use the identity

$$\frac{1}{2t}\partial_{\lambda_0}\ln\det(\tilde{I}+\widetilde{V}) = i\operatorname{tr}\hat{b}_{11}, \tag{9.20}$$

and after integrating of (9.19) with respect to λ_0 we obtain improved formula for Fredholm determinant. Of course, the integration constant may depend on t, therefore we need to use also the relations

$$\left(\partial_t - \frac{\lambda_0}{t}\partial_{\lambda_0}\right)\ln\det(\tilde{I}+\widetilde{V}) = i\operatorname{tr}\left(\hat{c}_{22}-\hat{c}_{11}\right), \tag{9.21}$$

and

$$\partial_t\hat{b}_{11} = \frac{1}{2t}\left(\partial_{\lambda_0}\hat{b}_{12}\cdot\hat{b}_{21} - \hat{b}_{12}\cdot\partial_{\lambda_0}\hat{b}_{21}\right). \tag{9.22}$$

These equations allow us to find integrating constant up to some function, which might depend on λ_0 since we did not specify $\hat{C}_0$ in (9.18). We skip the details; they are very similar to the free fermionic case spelled out in [29]. Thus, we have for the Fredholm determinant the following estimate,

$$\begin{aligned}\ln\det(\tilde{I}+\widetilde{V}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}&\left(|x-2\lambda t| + i\,\mathrm{sign}(\lambda_0-\lambda)\frac{d\psi(\lambda)}{d\lambda}\right)\\ &\times\ln\left\{1-\vartheta(\lambda)\left(1+e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\lambda_0)}\right)\right\}d\lambda\\ &-\frac{\nu^2}{2}\ln(2t) + C_0 + \mathcal{O}(\ln^2 t/t).\end{aligned} \tag{9.23}$$

Here C_0 includes integrating constant and other terms depending on functions ϕ_A, ϕ_D, the ratio λ_0 and the non-dynamical physical parameters h, c, and T.

In order to calculate the correlation function of local fields we need to find the combination

$$\mathcal{B} = \int_{-\infty}^{\infty} dudv\hat{b}_{12}(u,v) \cdot \det\left(\tilde{I} + \widetilde{V}\right). \tag{9.24}$$

The formulæ we obtained for the Fredholm determinant and the operator $\hat{b}_{12}$ allow us to find the asymptotic expression for this object

$$\begin{aligned}\mathcal{B} = {} & C(\phi_D,\phi_A)(2t)^{-\frac{(\nu+1)^2}{2}} e^{\psi(\lambda_0)-it\lambda_0^2} \\ & \times \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(|x-2\lambda t| + i\,\mathrm{sign}(\lambda_0-\lambda)\frac{d\psi(\lambda)}{d\lambda}\Big)\right. \\ & \left.\times \ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\lambda_0)}\Big)\Big\}\, d\lambda\right\}\Big(1+\mathcal{O}(\ln^2 t/t)\Big).\end{aligned} \tag{9.25}$$

Here we have extracted all dependence on t and function $\psi(\lambda)$. The factor $C(\phi_D,\phi_A)$ depends no λ_0, h, c, and T, and it does not depend on $\psi(\lambda)$; parameter ν is equal to

$$\nu = -\frac{1}{2\pi i}\ln\left[\Big(1-\vartheta_0 Z_0 e^{\phi_D(\lambda_0)}\Big)\Big(1-\vartheta_0 Z_0 e^{\phi_A(\lambda_0)}\Big)\right]. \tag{9.26}$$

and the function $\phi(\lambda)$ is defined by the equation,

$$\phi(\lambda) = \phi_A(\lambda) - \phi_D(\lambda). \tag{9.27}$$

In the case of positive chemical potential the calculations are quite similar. It is easy to show that the operators $\hat{B}_{12}^{(0)}$ and $\hat{B}_{21}^{(0)}$ (5.33) satisfy the same system of the differential equations (9.5), (9.6) as the operators $\hat{b}_{12}$ and $\hat{b}_{21}$. The asymptotic behavior of $\hat{B}_{12}^{(0)}$ and $\hat{B}_{21}^{(0)}$ was found in the previous section—Eq. (8.81). Thus one can use the same approach in order to obtain more precise estimates. We do not present here the details since in the section 11 we will develop a modified method allowing to obtain the most important, i.e. $\psi'(\lambda)$ - correction to the leading term of the asymptotics directly, without using the differential equations.

Finally, we need to say several words about the regularization introduced in the section 3. The only object in (9.25), which depends on ϵ, is the factor $C(\phi_D,\phi_A)$, since it depends on $\hat{\varrho}_0^{(c)}$ and $|\mathring{1}\rangle\langle\mathring{2}|$. The exponential and power law factors do not depend on the regularization. We shall see later that $C(\phi_D,\phi_A)$ is only responsible for a numerical factor in the asymptotics of the correlation function (1.1)

10 Modified approach. Preliminary discussions

Assuming the dual fields $\psi(\lambda)$, $\phi_A(\lambda)$ and $\phi_D(\lambda)$ to be classical functions, we found the asymptotic expression for $\mathcal{B}$, which is the combination of the Fredholm determinant $\det\Big(\tilde{I}+\widetilde{V}\Big)$ and factor $\hat{b}_{12}$ (2.18). This asymptotic expression has the form

$$\mathcal{B} = \mathcal{F}_c(\phi_A, \phi_D, \psi) t^{\mathcal{F}_p(\phi_A,\phi_D)} e^{t\mathcal{F}_e(\phi_A,\phi_D)} \Big(1 + \mathcal{O}(\ln^2 t/t)\Big), \tag{10.1}$$

where $\mathcal{F}_c$, $\mathcal{F}_p$ and $\mathcal{F}_e$ are functionals of the dual fields. Their representations are given in (9.25). The functionals $\mathcal{F}_e\Big(\phi_A, \phi_D\Big)$ and $\mathcal{F}_p\Big(\phi_A, \phi_D\Big)$, defining the exponential and power laws respectively, are found explicitly

$$t\mathcal{F}_e(\phi_A, \phi_D) = -it\lambda_0^2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} |x - 2\lambda t| \ln\Big\{1 - \vartheta(\lambda)\Big(1 + e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\lambda_0)}\Big)\Big\}\, d\lambda, \tag{10.2}$$

$$\mathcal{F}_p\Big(\phi_A, \phi_D\Big) = -\frac{1}{2}(\nu+1)^2, \tag{10.3}$$

where ν is given by (9.26). The functional $\mathcal{F}_c$ depends only on the ratio $\lambda_0 = x/2t$, which is fixed, hence it behaves as a constant when t goes to infinity. This functional consists of two parts, which play essentially different roles in the process of averaging with respect to auxiliary vacuum. Let us extract from $\mathcal{F}_c$ the part, containing the field ψ, namely

$$\mathcal{F}_c = \mathcal{F}_{c\psi}(\phi_A, \phi_D, \psi)\mathcal{F}_{c\phi}(\phi_A, \phi_D), \tag{10.4}$$

where

$$\mathcal{F}_{c\psi}\Big(\phi_A, \phi_D, \psi\Big) = \psi(\lambda_0) + \frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{sign}(\lambda_0 - \lambda)\frac{d\psi(\lambda)}{d\lambda} \ln\Big\{1 - \vartheta(\lambda)\Big(1 + e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\lambda_0)}\Big)\Big\}\, d\lambda. \tag{10.5}$$

For the functional $\mathcal{F}_{c\phi}$, which does not depend on ψ, we do not have any explicit formula; this functional depends on the operator $\hat{\varrho}_0^{(c)}$. Thus, the asymptotic behavior of $\mathcal{B}$ is found up to the constant factor $\mathcal{F}_{c\phi}$. The similar factor in the free fermionic case can be determined by the use of the auxilary differential equations with respect to the temperature and chemical potential (cf. [29]). Here we do not pursue such goals, and we restrict ourselves by establishing the exponential and power asymptotic laws. Thus, in the case of the dual fields being classical functions, we have complited the analysis. It should be also mentioned that the exponential law which we found coincide with the one obtained in [4] via the independed and more direct methods.

However, as we have already mentioned in the introduction and section 2, the operator nature of the dual fields strongly changes the situation at the stage of calculation of the vacuum mean value $(0|\mathcal{B}|0)$. One must be sure that the corrections to the leading term do not contribute into it after the averaging. In this section we demonstrate that the result (9.25) is not satisfactory from this point of view.

Recall the commutation relations between creation and annihilation parts of the dual fields (2.23)

$$\begin{aligned}
&[p_A(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_A(\mu)] = \ln h(\mu,\lambda),\\
&[p_D(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_D(\mu)] = \ln h(\lambda,\mu),\\
&[p_\psi(\lambda), q_\psi(\mu)] = \ln[h(\lambda,\mu)h(\mu,\lambda)],
\end{aligned} \tag{10.6}$$

where $h(\lambda,\mu) = (\lambda-\mu+ic)/ic$. It is important that operators p_A, p_D, q_A and q_D commute with each other:

$$[p_j(\lambda), q_k(\mu)] = 0, \qquad j,k \in \{A, D\}, \tag{10.7}$$

therefore the vacuum expectation value of any expression containing only the dual fields ϕ_A and ϕ_D is always trivial:

$$(0|\mathcal{F}(\phi_A,\phi_D)|0) = \mathcal{F}(0,0). \tag{10.8}$$

A non-trivial mean value arises if and only if the dual field ψ presents. That is why we paid so much attention to this dual field in the previous sections.

Consider the asymptotic representation (10.1) from the point of view of its averaging. It is easy to see that if we restrict ourselves with the exponential and power law factors, then we immediately obtain

$$(0|t^{\mathcal{F}_p(\phi_A,\phi_D)}e^{t\mathcal{F}_e(\phi_A,\phi_D)}|0) = t^{\mathcal{F}_p(0,0)}e^{t\mathcal{F}_e(0,0)}, \tag{10.9}$$

since the functionals $\mathcal{F}_p$ and $\mathcal{F}_e$ depend only on the fields ϕ_A and ϕ_D. Even if we take into consideration the functional $\mathcal{F}_{c\phi}$, then nothing will change for the exponential and power law factors because of the same reason—$\mathcal{F}_{c\phi}$ does not depend on ψ. We would like to emphasize especially that although the functional $\mathcal{F}_{c\phi}$ is unknown, this fact should not be considered as an essential deficiency. The fields ϕ_A and ϕ_D do not influence on the exponential and power laws factors in the process of averaging.

At the same time the presence of the functional $\mathcal{F}_{c\psi}\Big(\phi_A,\phi_D,\psi\Big)$ completely changes the situation. It depends on the dual field ψ, and it turns out that this operator changes the correlation radius, as well as the pre-exponent. Even the simplest factor $e^{\psi(\lambda_0)}$ shifts the arguments of $\mathcal{F}_{c\phi}$, $\mathcal{F}_p$ and $\mathcal{F}_e$ due to the evident property [5],

$$e^{p_\psi(\lambda_0)}\mathcal{F}\Big(\phi_A(\mu),\phi_D(\mu)\Big)|0) = \mathcal{F}\Big(\phi_A(\mu)+h(\mu,\lambda_0),\phi_D(\mu)+h(\lambda_0,\mu)\Big)|0). \tag{10.10}$$

The contribution of the remaining part of $\mathcal{F}_{c\psi}$ (10.5) is much more complicated, but it is clear that it changes the result (10.9).

In the present paper we are not going to study the procedure of averaging of the dual fields. This is done in [5], where the reader can find the details of the calculation of the vacuum expectation value. The main purpose of this paper is to present the asymptotic expression of the operator $\mathcal{B}$ in the form adjusted to the averaging procedure. The above considerations show that the constant

factor $\mathcal{F}_{c\psi}$ (i.e. the factor, which remains fixed while t goes to infinity) strongly changes the leading term of the asymptotics after calculation of the vacuum expectation value. It becomes clear now that the leading term of the asymptotics, which had been found in the sections 4 and 5, is not sufficient for the description of the correlation function.

One can ask now, whether we can restrict ourselves with the factor $\mathcal{F}_{c\psi}$? Can one guarantee that corrections $\mathcal{O}(\ln^2 t/t)$ do not give similar contribution into the vacuum mean value? Let us demonstrate that some corrections do provide such non-vanishing contribution.

In order to find corrections to the expression (9.25) one can use the differential equations (9.9) and (9.10). Substituting the asymptotic expansions (9.11), (9.12) into these equations we obtain an infinite set of the recurrent relations for the operators $\hat{\zeta}_{nk}$ and $\hat{\eta}_{nk}$. The last ones in turn provide us with complete asymptotic expansion of the Fredholm determinant and the operator $\hat{b}_{12}$. It is clear that corrections, which do not depend on ψ, can not change the leading exponential term $\exp\{t\mathcal{F}_e\}$ of the asymptotics after their averaging. Therefore we need to pay our attention mostly to the terms containing the field ψ.

Observe that $\hat{\zeta}_0(\lambda_0)$ and $\hat{\eta}_0(\lambda_0)$ are proportional to $e^{\psi(\lambda_0)}$ and $e^{-\psi(\lambda_0)}$ respectively. This suggests the following substitution,

$$\hat{\zeta}(\lambda_0,t)=e^{\psi(\lambda_0)}\hat{\hat{\zeta}}(\lambda_0,t),\qquad \hat{\eta}(\lambda_0,t)=e^{-\psi(\lambda_0)}\hat{\hat{\eta}}(\lambda_0,t). \tag{10.11}$$

into (9.9). We obtain for $\hat{\hat{\zeta}}$

$$-4i\frac{\partial\hat{\hat{\zeta}}}{\partial t}-\frac{4}{t}(\hat{\hat{\zeta}}\hat{\hat{\eta}}\hat{\hat{\zeta}}-i\nu\hat{\hat{\zeta}})=-\frac{1}{t^2}\Big\{\Big[(\nu')^2\ln^2(2t)-\nu''\ln(2t)-2\nu'\psi'\ln(2t)+\psi''+\psi'^2\Big]\hat{\hat{\zeta}}$$

$$-2\Big(\nu'\ln(2t)-\psi'\Big)\hat{\hat{\zeta}}'+\hat{\hat{\zeta}}''\Big\}. \tag{10.12}$$

Similar equation arises for the operator $\hat{\hat{\eta}}$. It is easy to see that the coefficient $\hat{\zeta}_{10}$ of the asymptotic expansion (9.11) contains the term

$$\hat{\hat{\zeta}}_{10}=\psi'^2(\lambda_0)\hat{\hat{\zeta}}_0+\dots\ , \tag{10.13}$$

and hence $\hat{b}_{12}$ contains the term $\frac{1}{t}\psi'^2(\lambda_0)\hat{\zeta}_0$. If t goes to infinity, then this term vanishes. However, after calculation of the vacuum expectation of ψ'^2/t together with the exponential factor $e^{t\mathcal{F}_e}$ it becomes proportional to the positive power of t. Indeed, the action of the operator ψ on functions of ϕ_A and ϕ_D is equivalent to the differentiating [5], therefore we have

$$(0|\frac{1}{t}\psi'^2(\lambda_0)e^{t\mathcal{F}_e(\phi_A,\phi_D)}|0)\sim te^{t\mathcal{F}_e(0,0)}+\dots\ . \tag{10.14}$$

Thus we see that the term ψ'^2/t, which was small before averaging and, hence, could be considered as a correction, becomes large after calculation of the vacuum expectation. Of course,

this term forms only pre-exponent, but does not change the correlation radius. However, it is not difficult to see that the higher corrections contain terms ψ'^4/t^2, ψ'^6/t^3 etc., which turn into t^2, t^3 etc. after the averaging. Generally, all the terms of type ψ^n/t^m, where $n > m$, give positive powers of t after their averaging together with the exponential factor. The series of positive powers of t sums up into the exponent, and as a result the correlation radius changes.

Thus, we conclude that in order to find correct asymptotic behavior of the correlation function it is necessary to have some information about the complete asymptotic expansion of the Fredholm determinant and the operator $\hat{b}_{12}$. At least one needs to know the structure of the ψ - dependence of the higher corrections. The leading term (9.25) itself is not sufficient. This expression correctly describes the asymptotics of the operator $\mathcal{B}$, but not the asymptotics of its vacuum expectation value.

11 Modified approach. The 'shifted' saddle point

We have shown in the previous section that, in fact, in the process of calculation of the vacuum expectation the dual field ψ is effectively proportional to t. This prompts an idea how to modify the method we used for the asymptotic solution of the operator-valued RHP. Recall the jump matrix (3.11) of the original RHP (a):

$$\begin{aligned}
\hat{G}_{11}(\lambda) &= \hat{\imath} - \vartheta(\lambda) Z(\lambda,\lambda) e^{\phi_D(\lambda)} |1\rangle\langle 1|;\\
\hat{G}_{12}(\lambda) &= 2\pi i(\vartheta(\lambda)-1) Z(\lambda,\lambda) e^{\psi(\lambda)+\tau(\lambda)} |1\rangle\langle 2|;\\
\hat{G}_{21}(\lambda) &= -\frac{i}{2\pi}\vartheta(\lambda) Z(\lambda,\lambda) e^{\phi_A(\lambda)+\phi_D(\lambda)-\psi(\lambda)-\tau(\lambda)} |2\rangle\langle 1|;\\
\hat{G}_{22}(\lambda) &= \hat{\imath} - \vartheta(\lambda) Z(\lambda,\lambda) e^{\phi_A(\lambda)} |2\rangle\langle 2|.
\end{aligned} \tag{11.1}$$

Observe that the field ψ enters this matrix only in the combination with the function $\tau(\lambda)$: $\psi(\lambda)+\tau(\lambda)$. All the asymptotic analysis of the operator-valued RHP was based on decomposition of the solution in the vicinity of the point $\lambda_0 = x/2t$. This value is the saddle point of the function $\tau(\lambda)$:

$$\left.\frac{d\tau(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} = 0. \tag{11.2}$$

As we have mentioned, one can consider ψ to be effectively proportional to t. Therefore instead of point λ_0 we can consider new saddle point Λ, which is defined as

$$\left.\frac{d}{d\lambda}(\tau(\lambda)+\psi(\lambda))\right|_{\lambda=\Lambda} = 0, \tag{11.3}$$

or equivalently

$$\Lambda = \lambda_0 + \frac{i}{2t}\psi'(\Lambda), \tag{11.4}$$

where prime means derivative with respect to the argument.

From the point of view of the classical asymptotic analysis this 'shift' of the saddle point does not make much sense. In fact, the difference between the original saddle point λ_0 and shifted one Λ is of order $1/t$ and it vanishes, when t goes to infinity. The formal solution of the equation (11.4) can be given by the asymptotic series with respect to $1/t$ (particular case of (11.6)):

$$\Lambda = \lambda_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{2t}\right)^n \frac{d^{n-1}}{d\lambda_0^{n-1}} \left[(\psi'(\lambda_0))^n\right]. \tag{11.5}$$

The replacement of λ_0 by Λ means that we have shifted the center of the asymptotic expansion by asymptotically small value. Using series (11.5) one can always re-expand any function $f(\Lambda)$ into the asymptotic series over $1/t$ with the center at the original point λ_0:

$$f(\Lambda) = f(\lambda_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{2t}\right)^n \frac{d^{n-1}}{d\lambda_0^{n-1}} \left[(\psi'(\lambda_0))^n f'(\lambda_0)\right]. \tag{11.6}$$

Here we have used the Lagrange's theorem (see, for instance, [51]). As a result, the 'new' asymptotics would of course coincide with the 'old' one.

Nevertheless this approach appears to be extremely useful when one recalls the role of the operator ψ in the process of averaging. Indeed, studying the asymptotic solution of the RHP for the large t, we automatically take into account the field ψ.

Let us turn back to the RHP (a) and consider the substitution (4.2)

$$\hat{\chi}(\lambda) = \hat{\Phi}(\lambda)\hat{\mathcal{R}}(\lambda), \qquad \hat{\mathcal{R}} = \begin{pmatrix} \hat{\varrho}(\lambda) & 0 \\ 0 & (\hat{\varrho}^T(\lambda))^{-1} \end{pmatrix}. \tag{11.7}$$

As before, the operator $\hat{\varrho}(\lambda)$ is the solution of the RHP (b), however now we replace λ_0 by Λ in the corresponding jump matrix:

$$\hat{\varrho}_-(\lambda) = \hat{\varrho}_+(\lambda) \left(\theta(\Lambda - \lambda)\hat{G}_{11} + \theta(\lambda - \Lambda)(\hat{G}_{22}^T)^{-1}\right), \quad \lambda \in R. \tag{11.8}$$

Then one should literally repeat all the constructions of the section 4, replacing everywhere λ_0 by Λ. Clearly that eventually we shall arrive at the following results for the coefficients of the asymptotic expansion of the determinant of $\hat{\varrho}$:

$$\Delta_0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\mu \,\mathrm{sign}(\Lambda - \mu) \ln\left(1 - \vartheta(\mu)\left(1 + e^{\phi(\mu)\,\mathrm{sign}(\mu - \Lambda)}\right)\right), \tag{11.9}$$

$$\Delta_1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mu d\mu \,\mathrm{sign}(\Lambda - \mu) \ln\left(1 - \vartheta(\mu)\left(1 + e^{\phi(\mu)\,\mathrm{sign}(\mu - \Lambda)}\right)\right). \tag{11.10}$$

In comparison with the old results (4.38), (4.39) of the section 4, the only difference is that λ_0 is replaced by Λ. Since $\Lambda - \lambda_0 \sim 1/t$, it is clear that new representations for Δ_j and the old ones are asymptotically equivalent.

It should be noted that the point Λ must not be necesserely real, so that the jump contour in (11.8) might be slightly deformed to the complex λ with the natural re-definition of the θ-functions. Also, the formulae (11.9) and (11.10) should be generally understood as (see (1.27)),

$$\Delta_0 = \frac{1}{2\pi i}\int_{-\infty}^{\Lambda} d\mu \ln\Big(1-\vartheta(\mu)\big(1+e^{-\phi(\mu)}\big)\Big) \tag{11.11}$$
$$-\frac{1}{2\pi i}\int_{\Lambda}^{\infty} d\mu \ln\Big(1-\vartheta(\mu)\big(1+e^{\phi(\mu)}\big)\Big),$$

and

$$\Delta_1 = \frac{1}{2\pi i}\int_{-\infty}^{\Lambda} \mu d\mu \ln\Big(1-\vartheta(\mu)\big(1+e^{-\phi(\mu)}\big)\Big) \tag{11.12}$$
$$-\frac{1}{2\pi i}\int_{\Lambda}^{\infty} \mu d\mu \ln\Big(1-\vartheta(\mu)\big(1+e^{\phi(\mu)}\big)\Big),$$

with the integrations go possibly in the complex plane. In all the following formulae we will always assume the same interpretation of the integrals involving the sign-functions.

In order to obtain the Fredholm determinant of the operator $\tilde{I}+\widetilde{V}$ one need to integrate (4.38) and (4.39) with respect to x and t respectively. Using

$$\partial_x \to \frac{1}{2t_s}\partial_\Lambda, \qquad \partial_t \to \partial_t - \frac{\Lambda}{t_s}\partial_\Lambda, \qquad \text{where} \qquad t_s = t - \frac{i}{2}\psi''(\Lambda), \tag{11.13}$$

we have

$$\begin{aligned}\ln\det(\tilde{I}+\tilde{V}) \;\approx\; & \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\Big(x-2\mu t+i\psi'(\mu)\Big)\,\mathrm{sign}(\Lambda-\mu)\\ & \times\ln\Big(1-\vartheta(\mu)\big(1+e^{\phi(\mu)\,\mathrm{sign}(\mu-\Lambda)}\big)\Big).\end{aligned} \tag{11.14}$$

It is worth mentioning that in the framework of our old approach we obtained the term $\psi'(\lambda)$ in the integrand (11.14) only after studying the localized RHP and differential equations. As we have mentioned already in the previous section, it is this term which is the most important for the calculation of the vacuum expectation value, and it is remarkable that the new approach allows us to obtain this term already at the first stage of the asymptotic analysis of the RHP. On the other hand, one should not be surprised by this fact. The new treatment of the saddle point (11.3) in fact means that we consider function ψ as it would be proportional to t. Therefor, it is quite natural that in addition to the term $x-2\mu t$ we have obtained $i\psi'$. One can also notice that

$$x - 2\lambda t + i\psi'(\lambda) = i(\tau(\lambda)+\psi(\lambda))', \tag{11.15}$$

and thus, the integrand in the Eq. (11.14) reflects the fact that the jump matrix in the original RHP (a) depends only on the combination $\tau(\lambda)+\psi(\lambda)$. Thus, we see that the method proposed

in this section permits one to trace the dependency of the Fredholm determinant on the field ψ automatically, together with the dependency on the time t.

We notice again, that in reality the function ψ does not depend on t, and it remains fixed when t goes to infinity. Hence the representations (11.14) and (4.41) are asymptotically equivalent. They are not equivalent only when we recall the quantum operator nature of the field $\psi(\lambda)$

The localized RHP can be considered in the same manner as it had been done previously. We would like to mention only one moment. The decomposition of the $\tau(\lambda)+\psi(\lambda)$ at the point $\lambda=\Lambda$ has the form

$$\tau(\lambda)+\psi(\lambda)=\tau(\Lambda)+\psi(\Lambda)+it_s(\lambda-\Lambda)^2+\mathcal{O}(\lambda-\Lambda)^3, \tag{11.16}$$

where $t_s=t-i\psi''(\Lambda)/2$ was introduced in (11.13). We see that variable t_s plays the role of the new time. Therefore it is quite natural to look for the asymptotic expansion of the solution of the RHP with respect to t_s, but not to t. Up to replacement $t\to t_s$ and $\lambda_0\to\Lambda$ the solution of the new localized RHP formally coincides with the solution of the old one, in particular (cf. (9.1), (9.2)),

$$\hat{b}_{12}\approx\frac{\tilde{\gamma}\nu}{i\sqrt{2\pi}}(2t_s)^{-\nu-1/2}e^{\psi(\Lambda)+\tau(\Lambda)}\,\hat{\varrho}^{(c)}_{0-}|\mathring{1}\rangle\langle\mathring{2}|\,\hat{\varrho}^{(c)T}_{0+}, \tag{11.17}$$

$$\hat{b}_{21}\approx-\frac{\sqrt{2\pi}}{\tilde{\gamma}}(2t_s)^{\nu-1/2}e^{-\psi(\Lambda)-\tau(\Lambda)}\,(\hat{\varrho}^{(c)T}_{0+})^{-1}|\mathring{2}\rangle\langle\mathring{1}|\,(\hat{\varrho}^{(c)}_{0-})^{-1}, \tag{11.18}$$

where $\tilde{\gamma}$ is equal to

$$\tilde{\gamma}=2\pi Z(\Lambda,\Lambda)(\vartheta(\Lambda)-1)\Gamma(\nu)e^{\frac{i\pi\nu}{2}+\frac{3i\pi}{4}}, \tag{11.19}$$

and

$$\nu\equiv\nu(\Lambda)=-\frac{1}{2\pi i}\ln\left[\left(1-\vartheta(\Lambda)Z(\Lambda,\Lambda)e^{\phi_D(\Lambda)}\right)\left(1-\vartheta(\Lambda)Z(\Lambda,\Lambda)e^{\phi_A(\Lambda)}\right)\right]. \tag{11.20}$$

Of course, now $\hat{\varrho}^{(c)}_0=\hat{\varrho}^{(c)}(\Lambda)$ and vectors $|\mathring{1}\rangle$, $|\mathring{2}\rangle$ etc. are associated with Λ, but not with λ_0.

Finally, using the differential equations for $\hat{b}_{12}$ and $\hat{b}_{21}$ we can find the corrections to the Eqs. (11.17), (11.18). The most important is the dependency of these corrections on the function ψ. Substituting

$$\hat{b}_{12}=(2t_s)^{-\nu(\Lambda)-1/2}e^{\tau(\Lambda)+\psi(\Lambda)}\hat{\tilde{\zeta}}(\Lambda,t_s), \tag{11.21}$$

$$\hat{b}_{21}=(2t_s)^{\nu(\Lambda)-1/2}e^{-\tau(\Lambda)-\psi(\Lambda)}\hat{\tilde{\eta}}(\Lambda,t_s), \tag{11.22}$$

into equations (9.5), (9.5), we obtain, for example, for $\hat{\tilde{\zeta}}$

$$-4i\frac{\partial\hat{\tilde{\zeta}}}{\partial t_s}-\frac{4}{t_s}(\hat{\tilde{\zeta}}\hat{\tilde{\eta}}\hat{\tilde{\zeta}}-i\nu\hat{\tilde{\zeta}})=-\frac{1}{t_s^2}\left\{\left[(\nu')^2\ln^2(2t)-\nu''\ln(2t)\right]\hat{\tilde{\zeta}}-2\nu'\hat{\tilde{\zeta}}'\ln(2t)+\hat{\tilde{\zeta}}''\right\}+\mathcal{O}(\ln(t_s)t_s^{-3}), \tag{11.23}$$

where prime now means the derivative with respect to Λ. Thus, up to the corrections of the t_s^{-3} order we again came back to the equation (9.9). However now we have already taken into account the factor $e^{\psi(\Lambda)}$, and hence we do not need to make a substitution similar to (10.11). Therefore,

the replacement of variables $(\lambda_0, t) \to (\Lambda, t_s)$ allows us to remove the dependency on the function ψ from the terms proportional to t_s^{-2}.

It is easy to see that the asymptotic expansion similar to (9.11) formally solves the Eq.(11.23)

$$\hat{\tilde{\zeta}} = \hat{\tilde{\zeta}}_0(\Lambda) + \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \hat{\tilde{\zeta}}_{nk}(\Lambda) \frac{(\ln(2t_s))^k}{t_s^n}, \tag{11.24}$$

One can write similar expansion for the operator $\hat{\tilde{\eta}}$ also.

In order to study the recurrence arising for the coefficients $\hat{\tilde{\zeta}}_{nk}$ and $\hat{\tilde{\eta}}_{nk}$, one needs to know the explicit expressions for the terms $\mathcal{O}(t_s^{-3})$ in (11.23). It is not difficult to find these terms, however, for our goal, it is enough to check that the terms of the order t_s^{-3} are linear with respect to ψ. In addition, equation (11.23) contains the terms of the order t_s^{-4}, which are quadratic with respect to ψ. It is clear that for given n, the coefficients $\hat{\tilde{\zeta}}_{nk}$ and $\hat{\tilde{\eta}}_{nk}$ are polynomials of ψ and its derivatives of the $(n-1)$ degree or less. In other words, in the framework of the modified approach any large time correction, as it is expected, has the form ψ^m/t_s^n, where $m < n$. Therefore, as we have explained in the previous section, these corrections remain small even after their averaging with respect to the auxiliary vacuum, and hence we can drop them out.

Summarizing the above considerations, we obtain, in the case of negative chemical potential, the following asymptotics for the operator $\mathcal{B}$,

$$\begin{aligned}\mathcal{B} &= C(\phi_D, \phi_A, \Lambda)(2t_s)^{-\frac{(\nu(\Lambda)+1)^2}{2}} e^{\psi(\Lambda)+\tau(\Lambda)} \\ &\times \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(x - 2\lambda t + i\psi'(\lambda)\Big)\,\mathrm{sign}(\Lambda-\lambda)\right. \\ &\left.\times \ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\,\mathrm{sign}(\lambda-\Lambda)}\Big)\Big\}\,d\lambda\right\}\Big(1+\mathcal{O}(\ln^2 t/t)\Big).\end{aligned} \tag{11.25}$$

In distinction to the result (9.25), this asymptotic equation takes into account all the corrections which make a contribution into the leading term after the averaging. Corrections to the expression (11.25) remain small after calculation of their vacuum expectation value.

Comparing the results (9.25) and (11.25) one can observe that in the last case the constant factor $C(\phi_D, \phi_A, \Lambda)$ depends on the shifted saddle point Λ, which in turn implicitly depends on ψ. As before this constant factor is not explicitly defined since it contains the operator $\hat{\varrho}_0^{(c)}$. In the framework of the old method $\hat{\varrho}_0^{(c)}$ depended only on the dual fields ϕ_A and ϕ_D, hence this operator did not influence the leading term of the asymptotics. However now $\hat{\varrho}_0^{(c)}$ implicitly depends on the field ψ, and it might change the asymptotics. Since we do not know explicit expression for the operator $\hat{\varrho}_0^{(c)}$, we might worry about calculation of its vacuum expectation value. However, it is easy to see that the contribution of the constant factor $C(\phi_D, \phi_A, \Lambda)$ remains constant after

averaging. Indeed, remembering the decomposition (11.6) we have

$$C(\phi,\Lambda)=C(\phi,\lambda_0)+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2t}\right)^n\frac{d^{n-1}}{d\lambda_0^{n-1}}\left[(\psi'(\lambda_0))^n C'(\phi,\lambda_0)\right]. \tag{11.26}$$

All the terms of this series behave as $(\psi/t)^n$. Their vacuum expectation values together with the exponential factor are some constants:

$$(0|\left(\frac{\psi}{t}\right)^n e^{t\mathcal{F}_e(\phi_A,\phi_D)}|0)\sim e^{t\mathcal{F}_e(0,0)}(\mathcal{O}(1)+\ldots),\qquad t\to\infty. \tag{11.27}$$

Thus the replacement λ_0 by Λ in the functional C does not change the exponential and power laws of the asymptotics of the vacuum mean value, but changes only the common constant factor. Since in this paper we do not analyse this factor, we can simply replace the new saddle point by the old one in C. Similar arguments allow us to replace t_s by t in the power law factor. However we should keep the shifted saddle point Λ in the exponential factor (i.e. functions $\operatorname{sign}(\Lambda-\lambda)$ and $\tau(\Lambda)+\psi(\Lambda)$), and the power $\nu(\Lambda)$ since all these objects enter the result together with t and $\ln t$ respectively.

Our final answer in the case $h<0$ can be represented as follows,

$$\begin{aligned}\mathcal{B}&=C(\phi_D,\phi_A,\lambda_0)(2t)^{-\frac{(\nu(\Lambda)+1)^2}{2}}e^{\psi(\Lambda)+\tau(\Lambda)}\\&\times\exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(x-2\lambda t+i\psi'(\lambda)\Big)\operatorname{sign}(\Lambda-\lambda)\right.\\&\left.\times\ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\operatorname{sign}(\lambda-\Lambda)}\Big)\Big\}\,d\lambda\right\}\Big(1+\mathcal{O}(\ln^2 t/t)\Big).\end{aligned} \tag{11.28}$$

In the case of positive chemical potential the situation is quite similar. The asymptotics of $\mathcal{B}$ is given by the equation

$$\begin{aligned}\mathcal{B}&=\tilde{C}(\phi_D,\phi_A,\lambda_0)(2t)^{-\frac{\nu^2(\Lambda)}{2}}e^{\psi(\Lambda_1)+\tau(\Lambda_1)}\\&\times\exp\left\{\frac{1}{2\pi}\int_{\Gamma}\Big(x-2\lambda t+i\psi'(\lambda)\Big)\operatorname{sign}(\Lambda-\lambda)\right.\\&\left.\times\ln\Big\{1-\vartheta(\lambda)\Big(1+e^{\phi(\lambda)\operatorname{sign}(\lambda-\Lambda)}\Big)\Big\}\,d\lambda\right\}\Big(1+\mathcal{O}(1/\sqrt{t})\Big).\end{aligned} \tag{11.29}$$

The general structure of this result is the same as for negative chemical potential; namely, we have the exponential law, the power law, and the constant factors. However, all these factors slightly differ from the ones in (11.28). First, the integral in the exponent is taken along the contour Γ (see Fig.2). Secondly, we have the sum $\tau(\Lambda_1)+\psi(\Lambda_1)$ instead of $\tau(\Lambda)+\psi(\Lambda)$. Thirdly, the pre-exponent

powers of t also are different. The reasons of these difference were explained in the sections 3 and 5.

In conclusion we would like to touch briefly the free fermionic limit. In this limit the coupling constant c goes to infinity, and one can put all dual fields equal to zero. In this case Eqs. (11.28), (11.29) exactly reproduce the results of [29].

Summary

We study the operator-valued RHP (3.1), (3.11). This RHP allows to find the large time and long distance asymptotics of the Fredholm determinant, which in turn describes the temperature correlation function of the local fields of the QNLS model out off free-fermionic point. Although we use the advanced technique based on the nonlinear steepest descent method [38], the main idea of the scheme applied in the present paper mostly coincides with the approach considered in [29] for the free fermionic limit of the model. However, as we saw, the operator-valued RHP is essentially more complicated, and not only from the technical point of view. In particular, the solution of the RHP was given up to the function $\hat{\varrho}(\lambda, u, v)$, which is defined as a solution of a certain integral equation.. At the same time, in spite of this incompleteness, we were able to obtain explicit expressions for the exponential and power laws of the operator $\mathcal{B}$.

A special problem arises because of the dual fields. Due to the presence of these quantum operators one needs to take care not only of the asymptotics of the operator $\mathcal{B}$ itself, but also of the asymptotics of its vacuum expectation value. This leads us to the non-standard asymptotic analysis, which includes the shift of the saddle point.

The remaining stage in the calculation of the correlation function is the averaging of the results (11.28), (11.29) with respect to the auxiliary vacuum. The method for this averaging was developed in [5], although in that paper the contribution of some corrections were not considered. Nevertheless, one can definitely state that the calculation of the vacuum expectation value leads to nonlinear integral equations, closely related to the equations of the Thermodynamic Bethe Anzats. The final result for the asymptotics of the correlation function can be formulated in terms of the solutions of these equations. We shall consider these questions in the forthcoming publications.

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A The Fredholm determinant

The complete determinant representation for the correlation function (1.1) has the form

$$\langle\Psi(0,0)\Psi^\dagger(x,t)\rangle_T = -\frac{e^{-iht}}{2\pi}(0|\frac{\det\left(\tilde I+\widetilde V\right)}{\det\left(\tilde I-\frac{1}{2\pi}\widetilde K_T\right)}\int_{-\infty}^{\infty}\hat b_{12}(u,v)dudv|0). \tag{A.1}$$

The integral operator $\tilde I+\widetilde V$ acts on the real axis as

$$\left(\tilde I+\widetilde V\right)\circ f(\mu)=f(\lambda)+\int_{-\infty}^{\infty}\widetilde V(\lambda,\mu)f(\mu)d\mu, \tag{A.2}$$

where $f(\lambda)$ is some trial function. The kernel $\widetilde V(\lambda,\mu)$ is equal to

$$\widetilde V(\lambda,\mu)=\frac{1}{\lambda-\mu}\int_{-\infty}^{\infty}du(E_+(\lambda|u)E_-(\mu|u)-E_-(\lambda|u)E_+(\mu|u)). \tag{A.3}$$

Here

$$E_+(\lambda|u)=\frac{1}{2\pi}\frac{Z(u,\lambda)}{Z(u,u)}\left(\frac{e^{-\phi_A(u)}}{u-\lambda+i0}+\frac{e^{-\phi_D(u)}}{u-\lambda-i0}\right)\sqrt{\vartheta(\lambda)}$$
$$\times e^{\psi(u)+\tau(u)+\frac12(\phi_D(\lambda)+\phi_A(\lambda)-\psi(\lambda)-\tau(\lambda))}, \tag{A.4}$$

$$E_-(\lambda|u)=\frac{1}{2\pi}Z(u,\lambda)e^{\frac12(\phi_D(\lambda)+\phi_A(\lambda)-\psi(\lambda)-\tau(\lambda))}\sqrt{\vartheta(\lambda)}, \tag{A.5}$$

The representation (A.1) contains also one more Fredholm determinant, however the last one does not depend on time t and distance x, as well as on dual fields:

$$\widetilde K_T(\lambda,\mu)=\frac{2c}{(\lambda-\mu)^2+c^2}\sqrt{\vartheta(\lambda)\vartheta(\mu)}. \tag{A.6}$$

Starting from the kernel $\widetilde V(\lambda,\mu)$ one can construct the jump matrix of the operator-valued RHP, using the standard procedure [3], [8]

$$\hat G^{in}(\lambda)=\hat I+2\pi i\begin{pmatrix}-E_+(\lambda|u)E_-(\lambda|v) & E_+(\lambda|u)E_+(\lambda|v)\\ \\ -E_-(\lambda|u)E_-(\lambda|v) & E_-(\lambda|u)E_+(\lambda|v)\end{pmatrix}. \tag{A.7}$$

In order to obtain the jump matrix (3.2), one need to make the following transformation

$$\hat G(\lambda)=\hat\chi^0_+(\lambda)\hat G^{in}(\lambda)\left(\hat\chi^0_-(\lambda)\right)^{-1}, \tag{A.8}$$

where

$$\hat\chi^0(\lambda|u,v)=\hat I+\frac{e^{\psi(u)+\tau(u)}}{\lambda-u}\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{A.9}$$

After this transformation we arrive at the RHP (a), considered in the section 3.

B The regularized integral operator

The regularization of the RHP considered in the section 3 corresponds to new Fredholm determinant $\det(\tilde{I}+\widetilde{V}_\epsilon)$. The kernel $\widetilde{V}_\epsilon(\lambda,\mu)$ of new integral operator has just the same form as the original one

$$\widetilde{V}_\epsilon(\lambda,\mu)=\frac{1}{\lambda-\mu}\int_{-\infty}^{\infty}du(E_+^\epsilon(\lambda|u)E_-^\epsilon(\mu|v)-E_-^\epsilon(\lambda|u)E_+^\epsilon(\mu|v)). \tag{B.1}$$

Here

$$E_-^\epsilon(\lambda|u)=\frac{1}{2\pi}Z(u,\lambda)Z(\lambda,\lambda)\sqrt{\frac{\vartheta(\lambda)}{\mathcal{N}_\epsilon(\lambda)}}e^{\frac{1}{2}(\phi_A(\lambda)+\phi_D(\lambda)-\psi(\lambda)-\tau(\lambda))}, \tag{B.2}$$

$$E_+^\epsilon(\lambda|u)=E^\epsilon(\lambda|u)E_-^\epsilon(\lambda|u), \tag{B.3}$$

and

$$E^\epsilon(\lambda|u)=\frac{1}{Z(u,\lambda)}\int_{-\infty}^{\infty}d\xi dw\frac{e^{\psi(\xi)+\tau(\xi)}}{\mathcal{N}_\epsilon(\xi)Z(\xi,\xi)}\delta_\epsilon(u-\xi)\delta_\epsilon(w-\xi)$$
$$\times Z(u,\xi)Z(w,\xi)Z(w,\lambda)\left[\frac{e^{-\phi_D(\xi)}}{\xi-\lambda-i0}+\frac{e^{-\phi_A(\xi)}}{\xi-\lambda+i0}\right]. \tag{B.4}$$

It is easy to see that the limit $\epsilon\to 0$ is well defined in all formulæ. In particular $\widetilde{V}_\epsilon\to\widetilde{V}$.

The original jump matrix $\hat{G}^{in}$ has just the same structure as the matrix (A.7)

$$\hat{G}^{in}(\lambda)=\hat{I}+2\pi i\begin{pmatrix}-E_+^\epsilon(\lambda|u)E_-^\epsilon(\lambda|v) & E_+^\epsilon(\lambda|u)E_+^\epsilon(\lambda|v)\\ \\ -E_-^\epsilon(\lambda|u)E_-^\epsilon(\lambda|v) & E_-^\epsilon(\lambda|u)E_+^\epsilon(\lambda|v)\end{pmatrix}. \tag{B.5}$$

In order to obtain the jump matrix (3.11) one need to make transformation similar to (A.8) with matrix

$$\hat{\chi}^{0\epsilon}(\lambda|u,v)=\hat{I}+\int_{-\infty}^{\infty}\frac{d\xi}{\lambda-\xi}\delta_\epsilon(u-\xi)\delta_\epsilon(v-\xi)Z(u,\xi)Z(v,\xi)e^{\psi(\xi)+\tau(\xi)}\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{B.6}$$

The expressions of the logarithmic derivatives of the regularized Fredholm determinant in terms of the coefficients $\hat{b}$ and $\hat{c}$ (2.19), (2.20) can be obtained in the same manner, as it had been done in [3]. The differential equations (2.21) also are obvious corollaries of the regularized RHP.

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